A dynamic programming approach for the valuation of callable corporate Bonds within the CIR framework

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Abstract

We show how to price corporate callable (res. putable) bonds in a model where interest rates are described by the Cox, Ingersoll and Ross model and default is described by a stochastic intensity model. Pricing is done by mean of a dynamic programming procedure, recently proposed for non-defaultable bonds, that is shown to be fast and accurate. The procedure is easy to implement and constitute an alternative to the use of similar precision but more elaborate difference elements methods. The interplay between default and callability is also discussed in the case of European options where closed form solutions are available.

1 Introduction

The valuation of corporate bonds is a non-trivial problem since it involves, at least, the simultaneous accounting of interest rate and credit risk. In addition, the majority of traded bonds embeds a call option that allows the issuer to pay the debt in advance (usually at par). Typically this call option is either of the American style or of the Bermudan type with a schedule similar to that of the coupon payments. When American/Bermudan type options are involved it is clearly not possible to price bonds other than with numerical techniques. Broadly speaking these fell into two main categories: partial differential equations (PDE's) based methods that use finite difference or more elaborate finite elements schemes and originate from the no-arbitrage approach, and backward numerical integration based methods (including trees and MonteCarlo methods) derived from the martingale approach. The numerical valuation of the embedded option is more demanding than it can appear at first sight, especially when the decision to call back the option should be taken in advance with respect to the effective call date, as it happens for most traded contracts (*e.g.* for U.S. Treasury bonds).

In this work we perform the pricing of defaultable callable bonds in a model with stochastic interest rates and an exogenous stochastic default process, using a fast and accurate dynamic programming procedure, recently proposed by Ben-Ameur *et al.*, to compute the option embedded in the bond. Our setting extends the results obtained by previous works, where the valuation was restricted either to pure defaultable bonds or to pure callable bonds. In addition, we will show how the call option *interacts* with default risk, thus modifying the naive decomposition of the contract even in the case of European options.

2 Related literature

Pricing of pure defaultable bonds dates back to the seminal work of Merton [Mer74], while pricing of callable bonds is usually traced back to Brennan and Schwartz [BS77] who used a standard finite differences approach. Our approach differs from both these works since we prefer to describe the default with an intensity model as done by Duffie *et al.* [DS99] and to use a dynamic programming approach as proposed by Ben-Ameur *et al.* [BABKL04] to price the embedded option. For the description of the dynamics of interest rates, we chose the model of Cox, Ingersoll and Ross [CIR85], that constitute a standard reference for both academics and practitioners. In this sense our work is the extension of that of Barone *et al.* [BBAC98] to the pricing of defaultable callable bonds.

While the use of an intensity model is motivated both by its simplicity and the fact that non-callable corporate bonds are relatively well priced on the European market within this model, the use of a dynamic programming approach is motivated from the fact that finite difference methods could be quite instable in this context. In fact, as reported in the classic textbook by Smith [Smi85] "very slowly decaying finite oscillations can occur with the Crank-Nicholson method in the neighbourhood of discontinuities in the initial values or between initial values and boundary values". Is not completely surprising then that first numerical results in the computation of embedded options were quite deceiving and in some cases even negative values were reported. Much later the work of Brennan and Schwartz, Hull and White proposed trinomial trees in the context of generalized versions of interest rate models with time-dependent parameters. Trees crudely approximate the dynamics of the underlying asset and convergence is much more difficult to keep under control. The issue of oscillations in the finite difference scheme was readdressed by Büttler and Waldvogel in 1996 [BW96] who suggested that finite differences could not be used to price callable bonds with advance notice because of the impact of discontinuities at notice dates; they proposed instead an approach based on Greens functions, that, however, are known only for a limited number of interest rate models. More recently, D'Halluin *et al.* (2001) [DFVG01] were able to stabilize the finite differences approach using finite elements, flux limiters and appropriate timestepping. As opposed to their approach, Ben-Ameur *et. al* [BABKL04] proposed a fast dynamic programming procedure that was claimed to be free of the aforementioned stability problems. In all the above works however the emphasis was put on the difficulty in computing the embedded option and the possibility of default of the issuer was neglected.

3 The CIR model

We assume the reader being familiar with the univariate version of the Cox, Ingersoll and Ross (hereafter CIR) model described in [CIR85]; in this section we will briefly recall those features that will be relevant for later computations. The notation adopted is essentially the one of De Felice and Moriconi [DFM91]. Under the natural probability measure $\mathbb{P}$ the dynamics of the spot rate $r(t)$ is:

$$dr(t) = \alpha\,[\gamma - r(t)]\,dt + \rho\,\sqrt{r(t)}\,dz(t) \qquad r(0) = r_0 \tag{1}$$

where α, γ, ρ are strictly positive, time-independent, parameters and $z(t)$ is a standard Brownian motion. The price of risk will be denoted by:

$$\pi(t, r(t)) = \frac{\pi}{\rho}\sqrt{r(t)}$$

so that under the risk neutral probability measure $\mathbb{Q}$ the drift term is given by:

$$\alpha\,[\gamma - r(t)] - [\rho\,\sqrt{r(t)}]\,[\frac{\pi}{\rho}\sqrt{r(t)}] = \alpha\,\gamma - (\alpha + \pi)r(t)$$

and the dynamics of the spot rate is:

$$dr(t) = \overline{\alpha}\,[\overline{\gamma} - r(t)]\,dt + \rho\,\sqrt{r(t)}\,dw(t) \qquad r(0) = r_0 \tag{2}$$

where $\overline{\alpha} = \alpha + \pi$, $\overline{\alpha}\,\overline{\gamma} = \alpha\gamma$ and $w(t)$ is a standard Brownian motion under $\mathbb{Q}$. Finally it is useful to consider the dynamics under the *forward risk adjusted* probability measure $\mathbb{F}_{\mathbb{Q}}$, which is obtained using the price of non-defaultable zero coupon bond $P(t,T) = A(t,T)\,\exp[-B(t,T)\,r(t)]$ as *numéraire*:

$$dr(t) = [\overline{\alpha}\,\overline{\gamma} - (\overline{\alpha} + \rho^2\,B(t,T))r(t)]\,dt + \rho\,\sqrt{r(t)}\,d\zeta(t) \qquad r(0) = r_0 \tag{3}$$

where $\zeta(t)$ is a standard Brownian motion under $\mathbb{F}_{\mathbb{Q}}$ and the functions $A(t,T)$ and $B(t,T)$ will be detailed later on in eq. (7).
As shown by Feller [Fel54], for each of the aforementioned probability measures the distribution of $r_T = r(T)$, conditioned to the observed value $r_t = r(t)$ is of the non-central χ^2 type, respectively:

1. under the natural probability measure $\mathbb{P}$

$$Prob^{\mathbb{P}}[r_T|r_t] \sim \chi^2(\delta_P\,r_T; \nu_P, \mu_P\,r_t)$$

$$\delta_P = \frac{4\alpha}{\rho^2\,(1 - e^{-\alpha\,\tau})} \quad \mu_P = \frac{4\alpha\,e^{-\alpha\,\tau}}{\rho^2\,(1 - e^{-\alpha\,\tau})} \quad \nu_P = 2\nu$$

2. under the *risk neutral* probability measure $\mathbb{Q}$

$$Prob^{\mathbb{Q}}[r_T|r_t] \sim \chi^2(\delta_Q\, r_T; \nu_Q, \mu_Q\, r_t)$$

$$\delta_Q = \frac{4\overline{\alpha}}{\rho^2\,(1-e^{-\overline{\alpha}\,\tau})} \quad \mu_Q = \frac{4\overline{\alpha}\,e^{-\overline{\alpha}\,\tau}}{\rho^2\,(1-e^{-\overline{\alpha}\,\tau})} \qquad \nu_Q = 2\nu$$

3. under the *forward risk adjusted* probability measure $\mathbb{F}_{\mathbb{Q}}$

$$Prob^{\mathbb{F}_{\mathbb{Q}}}[r_T|r_t] \sim \chi^2(\delta_F\, r_T; \nu_F, \mu_F\, r_t)$$

$$\delta_F = \frac{4}{\rho^2\,B(t,T)} \quad \mu_F = \frac{4\,d^2}{\rho^2}\frac{e^{d\tau}}{(e^{d\tau}-1)^2}\,B(t,T) \quad \nu_F = 2\nu$$

where d, ϕ, ν (the Brown-Dybvig parameters) and τ (time to maturity) are defined as

$$d = \sqrt{\overline{\alpha}^2 + 2\rho^2} \quad \phi = \tfrac{1}{2}(d+\overline{\alpha}) \quad \nu = \frac{2\alpha\gamma}{\rho^2} \quad \tau = T - t \tag{4}$$

and, as in the classic textbook of Johnson and Kotz [JK75], $f(x;\nu,\lambda)$ is the probability density function of a non-central χ^2 distribution with ν degrees of freedom and non-centrality parameter λ, whereas $F(x;\nu,\lambda)$ stands for the cumulative distribution.

4 Non-defaultable bonds and options

Let $v(t,T)$ be the discount factor from time T to time t, $P(t,T)$ be the price at time t of a riskless unitary zero coupon bond with maturity in T and $C(t,T,s)$ the price in t of a European call with maturity in T written on a riskless unitary zero coupon bond with maturity in s, with $s \geq T \geq t$. Denoting with $E_t^{\mathbb{M}}[\cdot]$ the expectation at time t according to the probability measure $\mathbb{M}$ and conditioned to the observed value r_t, one has

$$P(t,T) = v(t,T) = E_t^{\mathbb{Q}}[e^{-\int_t^T r(u)du}] = A(t,T)\,e^{-B(t,T)\,r_t} \tag{5}$$

$$\begin{aligned} C(t,T,s) &= E_t^{\mathbb{Q}}[e^{-\int_t^T r(u)du}\,\max\{P(T,s)-K,0\}] = \\ &= v(t,s)\,\chi^2(x_1;\nu_1,\mu_1) - K\,v(t,T)\,\chi^2(x_2;\nu_2,\mu_2) \end{aligned} \tag{6}$$

where:

$$A(t,T) = \left[\frac{d\,e^{\phi\tau}}{\phi\,(e^{d\tau}-1)+d}\right]^{\nu} \qquad B(t,T) = \frac{e^{d\tau}-1}{[\phi\,(e^{d\tau}-1)+d]} \tag{7}$$

and (cfr. [DFM91] p. 292 or [BM01] p. 56)

$$x_2 = 2\,r^\star\,[\underbrace{\frac{2d}{\rho^2[e^{d\tau}-1]}}_{a} + \underbrace{\frac{2\phi}{\rho^2}}_{b}] = \frac{4}{\rho^2}\frac{\phi\,(e^{d\tau}-1)+d}{e^{d\tau}-1}r^\star = \frac{4}{\rho^2}\frac{1}{B(t,T)}r^\star = \delta_x r^\star$$

$$x_1 = 2\,r^\star\,[\underbrace{\frac{2d}{\rho^2[e^{d\tau}-1]}}_{a} + \underbrace{\frac{2\phi}{\rho^2}}_{b} + B(T,s)] = \frac{4}{\rho^2}\frac{1}{B(t,T)}r^\star + 2B(T,s)r^\star = \frac{\delta_x + 2B(T,s)}{\delta_x}x_2$$

$$\nu_1 = \nu_2 = 2\nu \tag{8}$$

$$\mu_2 = 2\,\underbrace{\frac{4d^2}{\rho^4[e^{d\tau}-1]^2}}_{a^2}\,r_t e^{d\tau}\,\underbrace{\frac{\rho^2 B(t,T)}{2}}_{1/(a+b)} = \frac{4d^2}{\rho^2}\frac{e^{d\tau}}{[e^{d\tau}-1]^2}B(t,T)r_t$$

$$\mu_1 = 2\,\underbrace{\frac{4d^2}{\rho^4[e^{d\tau}-1]^2}}_{a^2}\,r_t e^{d\tau}\,\underbrace{\frac{2}{\delta_x + 2B(T,s)}}_{1/(a+b+B(T,s))} = \frac{\delta_x}{\delta_x + 2B(T,s)}\mu_2$$

Notice that $x_1 = \varphi_x x_2$ while $\nu_1 = \nu_2$ and $\mu_1 = \mu_2/\varphi_x$, with $\varphi_x = [\delta_x + 2B(T,s)]/\delta_x$, and that $r^\star$ is the value of $r(T)$ for which the option is at-the-money in T, *i.e.* the solution of the equation

$$A(T,s)\,e^{-r^\star B(T,s)} = K \tag{9}$$

It is useful to recall the proof of eq. (6) since it will be used later on for callable bonds. The result is rather straightforward when the *forward risk adjusted* measure is used. Firstly, one has that

$$\begin{aligned} C(t,T,s) = E_t^{\mathbb{Q}}[e^{-\int_t^T r(u)du}\ \max\{P(T,s)-K,0\}] = v(t,T)\,E_t^{\mathbb{F}_\mathbb{Q}}[\max\{P(T,s)-K,0\}] = \\ = v(t,T)\,\delta_F\int_0^{r^\star}[(A(T,s)e^{-B(T,s)r_T} - K)f^{\mathbb{F}_\mathbb{Q}}(\delta_F\,r_T;2\nu,\mu_F r_t)]\,dr_T = \\ = v(t,T)\,\delta_F\int_0^{r^\star}[A(T,s)e^{-B(T,s)r_T}f^{\mathbb{F}_\mathbb{Q}}(\delta_F\,r_T;2\nu,\mu_F r_t)]\,dr_T\ - v(t,T)\,K\,F(\delta_F r^\star;2\nu,\mu_F r_t) \end{aligned} \tag{10}$$

To solve the integral in eq. (10) it is necessary to prove the following two lemmas: [1]
Lemma 1

$$e^{-\beta x}f(x;\nu,\lambda) = E[e^{-\beta x}]\cdot\eta f(x\eta;\nu,\lambda/\eta) \tag{11}$$

*P*roof
The pdf of the χ^2 distribution can be written (cfr. [JK75], ch. 28, eq. 3):

$$f(x;\nu,\lambda) = \frac{e^{-\frac{1}{2}(x+\lambda)}}{2^{\nu/2}}\sum_{j=0}^{\infty}\frac{(\lambda x)^j x^{\nu/2-1}}{\Gamma(\nu/2+j)2^{2j}j!}$$

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[1] this relation is reported in the paper by Barone *et al.* [BBAC98] where (in our notation) it appears as $exp(-a\,x)\,f(\delta\,x;\nu,\mu) = E[exp(-a\,x)]f(x(\delta+2a);\nu,\mu\,\delta/(\delta+2a))$; however there is a factor $\delta+2a$, which is later reabsorbed in the cumulative function, that is missing.

Notice that term $(\lambda x)^j$ in the sum is invariant for transformations of the type $(x, \lambda) \to (\eta x, \lambda/\eta)$. Therefore:

$$f(x\eta;\nu,\lambda/\eta) = \frac{e^{-\frac{1}{2}(x\eta+\lambda/\eta)}}{2^{\nu/2}} \sum_{j=0}^{\infty} \frac{(\lambda x)^j (x\eta)^{\nu/2-1}}{\Gamma(\nu/2+j)2^{2j} j!} = \eta^{\nu/2-1} e^{-\frac{1}{2}(x\eta-x+\lambda/\eta-\lambda)} f(x;\nu,\lambda)$$

Moreover, since the Laplace transform $g(\beta;\nu,\lambda)$ of $f(x;\nu,\lambda)$ is

$$g(\beta;\nu,\lambda) = E[e^{-\beta x}] = \int_0^\infty e^{-\beta x} f(x;\nu,\lambda)dx = (1+2\beta)^{-\nu/2} e^{-\beta\lambda/(1+2\beta)}$$

choosing $\eta = (1+2\beta)$, one obtains

$$\begin{aligned} f(x\eta;\nu,\lambda/\eta) &= (1+2\beta)^{\nu/2-1}\, e^{-\frac{1}{2}[(1+2\beta-1)x+\lambda(\frac{1}{1+2\beta}-1)]} f(x;\nu,\lambda) = \\ &= (1+2\beta)^{\nu/2-1}\, e^{-\beta x} e^{\beta\lambda/(1+2\beta)} f(x;\nu,\lambda) = \frac{1}{1+2\beta} \frac{e^{-\beta x} f(x;\nu,\lambda)}{E[e^{-\beta x}]} \end{aligned}$$

and finally

$$e^{-\beta x} f(x;\nu,\lambda) = E[e^{-\beta x}] \cdot \eta f(x\eta;\nu,\lambda/\eta)$$

□

Lemma 2

$$\begin{aligned} \delta \int_0^{x^\star} e^{-\beta x} f(\delta x;\nu,\lambda)dx &= E^f[e^{-\beta x}] \cdot F(\varphi\, \delta x^\star;\nu,\lambda/\varphi) \\ E^f[e^{-\beta x}] = \delta \int_0^\infty f(\delta x;\nu,\lambda)\, e^{-\beta\, x} dx &= (\frac{\delta+2\beta}{\delta})^{-\nu/2}\, e^{-\beta\lambda/(\delta+2\beta)} = \varphi^{-\nu/2} e^{-\beta\lambda/(\varphi\delta)} \end{aligned} \tag{12}$$

*P*roof
It is sufficient to realize that defining $y = \delta x$ one has

$$\delta \int_0^\infty f(\delta x;\nu,\lambda)\, e^{-\beta\, x} dx = \int_0^\infty f(y;\nu,\lambda)\, e^{-\beta(y/\delta)} dy = E[e^{-(\beta/\delta)\, x}]$$

and then use Lemma 1 with $\varphi = (1+2\beta/\delta) = (\delta+2\beta)/\delta$, and finally integrate over x. □

The integral in eq. (10) can now be written [2]

$$\begin{aligned} &\delta_F \int_0^{r^\star} [A(T,s) e^{-B(T,s) r_T} f^{\mathbb{F}_\mathbb{Q}}(\delta_F\, r_T; 2\nu, \mu_F r_t)] dr_T = \\ &= E_t^{\mathbb{F}_\mathbb{Q}}[A(T,s) e^{-B(T,s)\, r_T}] \cdot F(\varphi_F\, \delta_F r^\star; 2\nu, \mu_F r_t/\varphi_F) = \\ &= F(\varphi_F\, \delta_F r^\star; 2\nu, \mu_F r_t/\varphi_F) \cdot \left\{ \begin{array}{ll} v(t,T,s) & \text{using the definition of } \mathbb{F}_\mathbb{Q} \\ A(T,s)\, \varphi_F^{-\nu}\, e^{-B(T,s)\mu_F r_t/(\varphi_F \delta_F)} & \text{using the Laplace transform} \end{array} \right. \end{aligned} \tag{13}$$

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[2] for a comparison one can use the expression of the moment generating function $\omega_{t,T}(u)$ as reported in [DFM91] p. 252 for the $\mathbb{P}$ measure, where (in their notation) $r(T) \sim \chi^2(2cr(T); 2\nu, 2b)$, or equivalently (in our notation) $r(T) \sim \chi^2(\delta_P r(T); \nu_P, \mu_P r(t))$:

$$\omega_{t,T}(u) = E_t[e^{-ur(T)}] = [1+(2u)/(2c)]^{-\nu}\, e^{-u(2b)/(2c+2u)} = (1+u/c)^{-\nu} e^{-b/(1+c/u)}$$

Notice that the last equality holds only if $u \neq 0$, while clearly $\omega_{t,T}(0) = 1$.

with $\varphi_F = [\delta_F + 2B(T,s)]/\delta_F$. *En passant* the two approches show that:

$$v(t,T,s) = \frac{v(t,s)}{v(t,T)} = A(T,s)\,\varphi_F^{-\nu}\, e^{-B(T,s)\mu_F r_t/(\varphi_F \delta_F)} \tag{14}$$

The final result is therefore

$$\begin{aligned} C(t,T,s) &= v(t,T)\,v(t,T,s)\,F(\varphi_F\,\delta_F r^\star; 2\nu, \mu_F r_t/\varphi_F) - v(t,T)\,K\,F(\delta_F r^\star; 2\nu, \mu_F r_t) = \\ &= v(t,s)F(\varphi_F\,\delta_F r^\star; 2\nu, \mu_F r_t/\varphi_F) - v(t,T)\,K\,F(\delta_F r^\star; 2\nu, \mu_F r_t) \end{aligned}$$

$\square$

that is the expression reported in the original CIR paper.

5 Non-defaultable callable bonds

For the valuation of bonds with embedded options we will use a dynamic programming technique proposed in [BABKL04]. The method works iteratively on a (*time*, *spot rate*) grid composed of $(n+\ell+1)\times(p+1)$ points, where the relevant times are the valuation date t_0, the n payment dates $\{t_1, t_2, \ldots, t_n = T\}$ and the ℓ embedded options exercise dates $\{\tau_1, \tau_2, \ldots, \tau_\ell\}$, all contractually known, while the $p+1$ spot rate values $\{r_0, r_1, \ldots, r_p\}$ are parameters of the method. Notice that we have assumed that the last coupon payment date t_n coincides with the maturity of the bond T; obviously this hypothesis can be trivially relaxed without relevant modifications.

For the sake of simplicity we shall considered a bond issued at time t_0 with maturity T and face value N, paying fixed coupons c with fixed periodicity Δt, so that $t_{k+1} - t_k = \Delta t$ for any $k = 0, 1, \ldots, n-1$. We shall further assume that the periodicity of the exercise dates is the same of that of the coupon dates, *i.e.* that $\tau_{k+1} - \tau_k = \Delta t$ for any $k = 1, \ldots, \ell - 1$, and that δ is the *notice period*, that is the interval between a given exercise date and the forthcoming coupon date. The interval between the emission and the first exercise date $\Delta = \tau_1 - t_0$ will be called the *protection period* (against early exercising). Finally we will assume that the strike prices for a callable (putable) bond $\{C_1, C_2, \ldots, C_\ell\}$ ($\{P_1, P_2, \ldots, P_\ell\}$) are contractually fixed and thus known at time t_0. Notice that we will not assume that the call (put) strike prices are all equal.

Let now $P(\tau_{m+1}, r)$ be the value of the bond with its embedded options as a function of the spot rate at time $t = \tau_{m+1}$ and $\widetilde{t_m} = \tau_m + \delta$ the coupon payment date in between τ_m and τ_{m+1}. At time $t = \tau_m$ the ex-coupon holding value of the bond $P^h(\tau_m, r_m)$ is:

$$P^h(\tau_m, r_m) = E^{\mathbb{Q}}_{\tau_m}\Big[e^{-\int_{\tau_m}^{\tau_{m+1}} r(u)\,du}\,P(\tau_{m+1}, r_{m+1})\Big] = v(\tau_m, \tau_{m+1})\,E^{\mathbb{F}_{\mathbb{Q}}}_{\tau_m}\Big[P(\tau_{m+1}, r_{m+1})\Big] \tag{15}$$

Since at τ_m the option can be exercised the value of the bond at τ_m will be

$$P(\tau_m, r_m) = \max\{P_m, \min\{C_m, P^h(\tau_m, r_m)\}\} + v(\tau_m, \widetilde{t}_m)\,c \tag{16}$$

which means that the bond is worth:

$$P(\tau_m, r_m) = \begin{cases} v(\tau_m, \widetilde{t}_m)(C_m + c) & \text{if the issuer calls} \\ P^h(\tau_m, r_m) + v(\tau_m, \widetilde{t}_m)c & \text{if the investor hold} \\ v(\tau_m, \widetilde{t}_m)(P_m + c) & \text{if the investor puts} \end{cases} \tag{17}$$

The two equations (15) and (16), together with the condition at maturity $P(T, r_i) = 1 + c$ ($i = 0, \ldots, p$), constitute the basis of the dynamic procedure and allow to price the bond.

Obviously the main difficulty lies in the computation of the integral in eq. (15). As shown in [BABKL04] this can be easily computed if $P(\tau_{m+1}, r)$ is approximated in a piecewise-linear fashion inside the interval $[r_0, r_{p+1}]$

$$P(\tau_{m+1}, r) = \sum_{i=0}^{p} (\alpha_i^{m+1} + \beta_i^{m+1} r)\, \mathbf{1}_{\mathbf{r_i \leq r < r_{i+1}}} \tag{18}$$

the expectation value is then given by the following sum

$$E_{\tau_m}^{\mathbb{F}_{\mathbb{Q}}}\Big[P(\tau_{m+1}, r_{m+1})\Big] = \sum_{i=0}^{p} \alpha_i^{m+1} I_{m,i} + \beta_i^{m+1} J_{m,i} \tag{19}$$

where

$$I_{m,i} = \delta_F \int_{r_i}^{r_{i+1}} f^{\mathbb{F}_{\mathbb{Q}}}(\delta_F r; 2\nu, \mu_F r_m)\, dr = F(\delta_F r_{i+1}; 2\nu, \mu_F r_m) - F(\delta_F r_i; 2\nu, \mu_F r_m) \tag{20}$$

$$J_{m,i} = \delta_F \int_{r_i}^{r_{i+1}} r\, f^{\mathbb{F}_{\mathbb{Q}}}(\delta_F r; 2\nu, \mu_F r_m)\, dr \tag{21}$$

While the first integral is trivial, the second one can be worked out explicitely by first writing:

$$f(x; \nu, \lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} h(x; \nu + 2j)$$

where $h(x; \nu)$ $(H(x; \nu))$ is the probability density (distribution) function of a chi-square distribution with ν degrees of freedom, and then taking advantage of the fact that

$$I_\ell^{(h)}(x, \nu) = \int x^\ell h(x; \nu) dx = -2^\ell\, \Gamma\Big(\ell + \frac{x}{2}, \frac{\nu}{2}\Big) \Big/ \Gamma\Big(\frac{\nu}{2}\Big)$$

where $\Gamma(z)$ is the Euler Gamma function defined as $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt$, $\gamma(a, z)$ is the (unnormalised) incomplete Gamma function defined as $\gamma(a, x) = \int_0^x e^{-t}\, t^{a-1}\, dt$ (as in [AS72] eq. 6.5.2) and $\Gamma(a, z) = \Gamma(a) - \gamma(a, z) = \int_z^\infty e^{-t}\, t^{a-1} dt$. (as in [AS72] eq. 6.5.3). Moreover noticing that

$$I_1^{(h)}(x, \nu) = -2\, \Gamma\Big(1 + \frac{\nu}{2}, \frac{x}{2}\Big) \Big/ \Gamma\Big(\frac{\nu}{2}\Big) = -\nu\, \Gamma\Big(\frac{2+\nu}{2}, \frac{x}{2}\Big) \Big/ \Gamma\Big(\frac{2+\nu}{2}\Big) = \nu I_0(x, \nu + 2) \tag{22}$$

One has

$$I_0^{(f)}(x, \nu, \lambda) = \int f(x; \nu, \lambda) dx = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} I_0^{(h)}(x, \nu) = -\sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} \frac{\Gamma\Big(\frac{\nu+2j}{2}, \frac{x}{2}\Big)}{\Gamma\Big(\frac{\nu+2j}{2}\Big)} \tag{23}$$

$$\begin{aligned} I_1^{(f)}(x, \nu, \lambda) &= \int x f(x; \nu, \lambda) dx = -\sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!}\, (\nu + 2j)\, \frac{\Gamma\Big(\frac{2+\nu+2j}{2}, \frac{x}{2}\Big)}{\Gamma\Big(\frac{2+\nu+2j}{2}\Big)} = \\ &= \nu\, I_0^{(f)}(x, \nu + 2, \lambda) - 2\Big(\frac{\lambda}{2}\Big) \sum_{j=1}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^{j-1}}{(j-1)!} \frac{\Gamma\Big(\frac{4+\nu+2(j-1)}{2}, \frac{x}{2}\Big)}{\Gamma\Big(\frac{4+\nu+2(j-1)}{2}\Big)} = \\ &= \nu\, I_0^{(f)}(x, \nu + 2, \lambda) + \lambda\, I_0^{(f)}(x, \nu + 4, \lambda) \end{aligned} \tag{24}$$

so that, finally [3]

$$J_{m,i} = \delta_F^{-1}\Big\{2\nu\,\Big[F(\delta_F\,r_{i+1}, 2\nu+2, \mu_F r_m) - F(\delta_F\,r_i, 2\nu+2, \mu_F r_m)\Big] + \\ + \lambda\Big[F(\delta_F r_{i+1}, 2\nu+4, \mu_F\,r_m) - F(\delta_F r_{i+1}, 2\nu+4, \mu_F\,r_m)\Big]\Big\} \qquad (25)$$

Notice that we have taken advantage of the fact that $F(x;\nu,\lambda) = I_0^{(f)}(x,\nu,\lambda)$ and that both $I_{m,i}$ and $J_{m,i}$ are expressed as functions of non-central chi-square distribution functions.

The set of three equations (19), (20) and (25) allows to compute the integral in eq. (15) and thus implement the procedure. Two issues are relevant. The first is that the *transition matrices* $I_{m,i}$ and $J_{m,i}$ can be computed only once if the exercise times are equally spaced, thus making the procedure extremely fast. The second is that equations (20) and (25) involve just the computation of the cumulative non-central chi square distribution. From the numerical point of view this computation is a bit delicate since it involves a sum over an infinite series. Several algorithms are available (see e.g. [Dyr04] for a review), some of which are asymptotic in spirit, like the well known one due to Sankaran [San63]. In fact it turns out that the choice of the best algorithm depends on the range of the parameters. In this work we shell make use of the Ding [Din92] algorithm, together with the Lanczos [Lan64] approximation of $\ln(\Gamma(x))$ for values of ν smaller than 2, and the Wiener Germ expansion [PR00] for larger values.

To illustrate the precision of the algorithm with a numerical example we have chosen the example reported in [BW96], also used in [DFVG01] and [BABKL04] to compare different methods. It refers to the Swiss Confederation bond CH0000157187 on the trading date Dec. 23, 1991, callable[4] annually in the last ten years of its lifetime, and whose relevant data are reported in table 1. Although irrelevant from the point of view of the dynamic programming method, we like to point out that the set of CIR parameters implies a value of ν smaller than 1, inconsistently with the assumption that spot rates should remain strictly positive.

For the computation we have used a grid of 1200 equally spaced points, with $r_0 = 0$ and $r_p = 3$. Our code is written in C, compiled with Dev-C^{++} 4.9.9.2 and runs in a Windows XP SP2 environment on a Pentium 4 notebook; typical computing time is of the order of 20 seconds. Table 2 summarizes the numerical results: it reports the values of the straight bond computed analytically (first column) and with the numerical procedure (second column); as one can see the agreement is at the level of about 0.1 basis points (third column) for the full range of r values considered (from 1% to 10%). Finally the fourth column reports the value of the callable bond, which is obviously lower than the corresponding straight bond. The values obtained here are in perfect agreement with what reported in [BABKL04].

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[3] In the paper by [BABKL04] the same integral is solved using the relation

$$\int_a^b x\,h(x;\nu)\,dx = \nu\,\big[H(b;\nu) - H(a;\nu)\big] - 2\,\big[b\,h(b;\nu) - a\,h(a;\nu)\big]$$

so that $J_{m,i}$ is given by:

$$J_{m,i} = \delta_F^{-1}\sum_{j=0}^{\infty} e^{-\lambda/2}\frac{(\lambda/2)^j}{j!}\Big\{(\nu+2j)[H(\delta_F\,r_{i+1};\nu+2j) - H(\delta_F\,r_i;\nu+2j)] + \\ - 2\,[r_{i+1}\,h(\delta_F\,r_{i+1};\nu+2j) - r_i\,h(\delta_F\,r_i;\nu+2j)]\Big\}$$

[4] in fact this bond has been called back on Feb. 25 2002

		CIR parameters		Year	Call price
ISIN	CH0000157187	α	0.54958046	$1-5$	1.000
Maturity T	20.172 years	ρ	0.38757496	6	1.005
Coupon c	4.25%	γ	0.0348468515	7	1.010
Principal	1.00	π	0.40663675	8	1.015
Notice period	0.1666 years			9	1.020
				10	1.025

Table 1: Input data for Example 1

$r(0)$	P_{CIR}	$P_{straight}$	$\delta(b.p.)$	$P_{callable}$
0.01	0.955246947936	0.955258258866	-0.113109	0.939212871880
0.02	0.931534875261	0.931545871961	-0.109967	0.915946593207
0.03	0.908451749557	0.908462440755	-0.106912	0.893296867134
0.04	0.885980609878	0.885991004063	-0.103942	0.871247077313
0.05	0.864104956302	0.864115061724	-0.101054	0.849781058842
0.06	0.842808737337	0.842818562019	-0.098247	0.828883085934
0.07	0.822076337679	0.822085889420	-0.095517	0.808537859933
0.08	0.801892566300	0.801901852682	-0.092864	0.788730497654
0.09	0.782242644862	0.782251673257	-0.090284	0.769446520048
0.10	0.763112196455	0.763120974031	-0.087776	0.750671841170

Table 2: Numerical results for example 1.

6 Defaultable bonds and options

To extend the results of the previous sections to defaultable bond, and similarly to Duffie *et. al* [DS99] and Barone *et al.* [BBAC98], we shall further assume that:

1. the state variables determining the values of the underlying asset and the state variables determining the occurrence of defaults and the fractional recovery thereof are independent;
2. the issuers default is supposed to be an unpredictable event described by a Poisson process with a risk-neutral (annual) constant hazard rate equal to h;
3. at the default time τ, the market value of the claim is reduced by a fractional loss equal to $L = 1 - rr$ (rr being the recovery rate) and the residual value is immediately paid by the creditor.

In fact our approach is based on one of the simplest stochastic default intensity models one can possibly build. Despite its simplicity the majority of non-callable corporate bond traded on the European market are in fact relatively well priced within the model. It is therefore natural to extend the approach to the case of callable bonds. To do that let's start by recalling the most relevant features of the model when applied to the pricing of bonds and European options; for more details we refer to [BABKL04].

Let $\mathbf{1}_{t\geq\tau}$ the indicator function of the event *default* at time τ. By virtue of independence one has:

$$E_t^{\mathbb{Q}}[\,\mathbf{1}_{T\leq\tau}\,X] = E_t^{\mathbb{Q}}[\mathbf{1}_{T\leq\tau}]\,E_t^{\mathbb{Q}}[X] = e^{-\int_t^T h(u)du}\,E_t^{\mathbb{Q}}[X] = e^{-h(T-t)}E_t[X] \qquad (26)$$

The above relation simply states that in the risk-neutral world the value of any defautable contingent claim is weighted by the survival probability. In case of default however, one has to make some hypothesis on the recovery mechanism. According to our assumptions a fraction of the pre-default market value is recovered, so that the value $V_D(t)$ at time t of the claim with promised payoff $V(T)$ can be written as

$$V_D(t) = E_t^{\mathbb{Q}}[\mathbf{1}_{T<\tau}\, e^{-\int_t^T r(u)du}\, V(T)] + \mathbf{1}_{T\geq\tau}\, L\, V_D(\tau^-) \tag{27}$$

The advantage of the fractional recovery rate assumption is that $V_D(t)$ can be computed under the risk-neutral probability $\mathbb{Q}$ as a default-free claim, just by discounting at the effective rate [5] $R(t) = r(t) + \eta = r(t) + hL$, that is:

$$V_D(t) = E_t^{\mathbb{Q}}\Big[e^{-\int_t^T R(u)du}\, V(T)\Big] = E_t^{\mathbb{Q}}\Big[e^{-\int_t^T (r(u)+\eta)du}\, V(T)\Big] = e^{-\eta(T-t)}V(t) \tag{28}$$

In principle Eq. (26) and (28) allow to write analytic expressions for the price $P_D(t,T)$ of a defaultable unitary zero coupon bond with maturity T and for an European option $C_D(t,T,s)$ with maturity T, written on a defaultable unitary zero coupon bond with maturity s exactly in the same fashion as for the non-defaultable case. However the hypothesis of fractional recovery for an option is somehow tricky. In fact whereas it is clear that for the price of a defaultable bond one has

$$P_D(t,T) = e^{-\eta(T-t)}P(t,T) \tag{29}$$

for that of a call option on a defaultable bond one often encounters in the literature (*e.g.* in[BBAC98]) the following expression

$$\begin{aligned} C_D(t,T,s) &= E_t^{\mathbb{Q}}[e^{-\int_t^T r(u)du}\, \mathbf{1}_{T\leq\tau}\, \max\{P_D(T,s) - K, 0\}] = \\ &= e^{-h(T-t)}v(t,T)\Big\{P_D(T,s)F(\varphi_F\,\delta_F r^*; 2\nu, \mu_F r_t/\varphi_F) - K\, F(\delta_F r^*; 2\nu, \mu_F r_t)\Big\} \end{aligned} \tag{30}$$

where r^* is the solution of the equation:

$$e^{-\eta(T-t)}A(t,T)e^{-B(t,T)\, r^*} = K$$

Notice that in the limit of full recovery in case of default ($L = 0$ and $\eta = 0$) the value of the pure defaultable bond coincides with the value of the straight bond, as expected. However, the value of the European call, as written in eq. (30) does not coincide with the value of the corresponding option written on a non defaultable bond. This is due to the fact that such expression has been derived with the assumption that if the underlying bonds issuer defaults before expiration then the option expires worthless. Therefore its value is decreased by a factor $\exp[-h(T-t)]$ which is just the risk neutral probability that there is no default before the option expiration date. In this sense while $P_D(t,T)$ is sensitive to η alone, while $C_D(t,T,s)$ is sensitive to η and h separately.

Differently, one could have assumed that if bankruptcy occurs at any time before expiration, then the holder (the issuer) can immediately exercise the option. In this case however the option is no longer a European option (since it can be exercised before maturity) but a kind of digital option triggered by the default event. As a consequence its value is increased. Clearly if the fractional loss is large, it is then likely that the option is out of the money at default, and thus that this extra component of the option value is null.

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[5] usually an idiosyncratic "liquidity premium" ℓ is also included, so that $\eta = hL + \ell$

7 Defaultable callable bonds

While one could expect to know contractually if the option can be exercised at default when the latter is written by a third party on a non-callable defaultable bond, when the writer is the issuer itself, and then the option is embedded in the bond, the situation is a little bit different since there is a *nettting* between the value of the non-callable component and that of the option operated by the issuer at emission. We will assume that such netting is implicitly done also at default time, that is that in the case of default the buyer recovers a fraction of the market value of the non-callable bond component in the instant preceding the default, netted by the same fraction of the value of the option at the instant before default. Notice that this means that if the option was in the money before the default event (which is likely to be) its value at default will be different from zero. Thus, in case of default, the issuer is allowed to refund the buyer a smaller amount. On the other hand, since this mechanism increases the value of the option, it decreases the value of the callable bond at emission.

The valuation is performed with the dynamic programming procedure, modified to take into account the possibility of default. The due modifications are trivial by virtue of eq. (28). However, we stress again that in this way the recovery of market value is effective also on the option component, because this is embedded in the value of the bond.

To clarify this effect with a numerical example we considered the a German Bank fixed rate coupon bond, ISIN DE0005217780, that expires on May, 29 2022 and is callable on May, 29 2012. At the valuation date, namely March, 31 2005, the bond issuer was rated as A3 by Moody's and its market value[6] was 116.6390 Euros.

This particular bond has been chosen because the value of the embedded option is a relevant fraction of the bond value (roughly 10%) and the rating of the issuer relatively low. To compute the bond value we firstly deduced the CIR parameters by fitting the EURIBOR and IRS values; then computed the individual $\eta(r)$ coefficients for a set of about 700 fixed rate non-callable coupon bonds of the financial sector, denominated in Euros and identified by the rating r, and finally smoothed the dependence from the rating with a three parameter (α_0, α_1, q) function $f(r) = \exp\{\alpha_0 + \alpha_1 r^q\}$. The details of this procedure are relevant for a numerical test of the model but not for the numerical example. Therefore we will limit us to report the input data for the computation as summarised in table 3, while table 4 shows the results of the computations where the principal of the bond has been scaled down to 1.

As one can notice the value of the straight bond and that of the pure callable bond are computed with an error of 0.02 basis points. The value of the embedded call option for the non-defaultable bond is also in agreement with the analytic expression at the same level of precision. Similarly, the value of the embedded option for the defaultable bond is in agreement with the analytic expression when the recovery rate is null, while (last line of table 4) it is larger by about 20 basis points when the recovery rate is 50%. Notice that we are keeping the value of η fixed, so that the value of the bond does not change, while $h(rr) = \eta/(1-rr)$ doubles moving from the case $rr = 0$ to the one $rr = 50\%$ and the corresponding analytic value of the call is decreased by a factor $\exp[h(0)\,(T-t)] = \exp[\eta\,(T-t)]$.

Finally, taking into account the accrued interests of 0.05518326, the estimated value of the bond is 116.0566, in remarkable agreement with what is quoted on the market (considering that we have used an average value of η).

As an example of a Bermudan type bond we considered a similar one, ISIN DE000EH0A2K6, that expires on Nov., 11 2016 and is callable yearly with a 5 days notice period. At the valuation

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[6] source Bloomberg

date, namely Dec., 31 2005, the bond issuer was rated as A3 by Moody's and its market value[7] was 99.301 Euros. As for the previous example the relevant input data are reported in table 5 and the numerical results summarised in table 6. Taking into account the accrued interest 0.4712328760 euros the final price is estimated to be 99.7077658576 in good agreement with the market value.

		parameters		Year	Call price
ISIN	DE0005217780	r	0.024964	7.16389	1.000
Maturity T	17.16389 years	d	0.245850		
Coupon c	6.6%	ϕ	0.241420		
Principal	1.00	ν	8.958545		
Notice period	-				
		η	0.0029746		

Table 3: Input data for Example 2

Straight bond			
$P_{CIR}^{straight}$	P_{DP}	δ(b.p.)	
1.353812523054	1.353809650325	0.028727	
Pure callable			
$P_{CIR}^{straight}$	P_{DP}	δ(b.p.)	C_{CIR}(b.p)
1.353812523054	1.236791443584	1170.210795	1170.208554
Pure defaultable			
$P_{CIR}^{defaultable}$	P_{DP}	δ(b.p.)	
1.310472653991	1.310469916031	0.027380	
Callable defaultable $rr = 0\%$			
$P_{CIR}^{defaultable}$	P_{DP}	δ(b.p.)	C_{CIR}(b.p.)
1.310472653991	1.215749026576	947.236274	947.229463
Callable defaultable $rr = 50\%$			
$P_{CIR}^{defaultable}$	P_{DP}	δ(b.p.)	C_{CIR}(b.p.)
1.310472653991	1.215749026576	947.236274	927.258006

Table 4: Numerical results for Example 2

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[7] source Bloomberg

		parameters		Year	Call price
ISIN	DE000EH0A2K6	r	0.024964	coupon dates	1.000
Maturity T	10.88219178 years	d	0.332739		
Coupon c	4.0%	ϕ	0.328389		
Principal	1.00	ν	8.928125		
Notice period	5 days				
		η	0.0028117		

Table 5: Input data for Example 3

	Straight bond	
$P_{CIR}^{straight}$	P_{DP}	δ(b.p.)
1.046062566563	1.046082253995	-0.196874
	Pure defaultable	
$P_{CIR}^{defaultable}$	P_{DP}	δ(b.p.)
1.019911544864	1.019930659459	-0.191146
	Callable defaultable	
$P_{CIR}^{defaultable}$	P_{DP}	δ(b.p.)
1.019911544864	1.001789987336	181.215575

Table 6: Numerical results for Example 3

8 Conclusions

We have shown how to implement the pricing of defaultable callable bonds in the CIR framework with constant intensity default rates. The pricing of the embedded options is performed with a new dynamic programming procedure, developed for non-defaultable bonds, which is both fast and precise. The comparison between the value computed with the procedure and the closed form results for bonds embedding an European option has been used not only to show the numerical accuracy of the method but also to put in evidence the interplay between default and callability. Having been extended to include the possibility of default, the model is now ready to be used in the empirical context for the valuation of corporate bonds.

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