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On the problem of feedback linearization

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Abstract

The paper studies and solves in a geometric framework the problem of partial feedback linearization for discrete-time dynamics. An algorithm for computing the largest linearizable subsystem is proposed. This approach can be considered as dual to the one already proposed in literature in an algebraic context. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The problem of linearization via regular static state feedback has been largely studied [6–8, 11, 13]. For continuous-time dynamics, an algorithm for computing the largest linearizable sub-system has been given in [9]. In discrete time the problem has been solved in [1] using the algebraic framework set in [4, 5], under the assumption of accessibility of the system. In the present paper the study is performed with reference to submersive nonlinear discrete-time dynamics in the geometric framework developed in [10, 11]. Such a geometric context is based on the introduction of an infinite set of canonical vector fields which are independent of the control variables and associated with the controllability directions. Through the analogies of tools and concepts, such a geometric approach enables us in many cases to unify the study of continuous-time, discrete-time and sampled dynamics. Presently, with reference to the problem of partial linearization, an algorithm for the computation of the largest linearizable subsystem is proposed following the lines of the continuous-time approach. It is based on the introduction of two sets of distributions which characterize the largest linearizable part and are defined in terms of the previously mentioned canonical vector fields.

Even if this common differential geometric framework leads to the statement of the results in a comparable way, this should not hide the specific difficulties occurring in discrete time. In fact, with reference to the difficulties encountered one concludes that continuous-time affine systems are very particular and should be compared to a class of discrete-time systems as discussed in [11] and pointed out in Section 3. As a result of the nonlinear control dependency of the dynamics, the extension is not straightforward and the results here proposed in discrete time can also be applied to the general nonlinear continuous-time case.

The present algorithm should be interpreted as the dual version of the one proposed in [1] where under some accessibility conditions the results are set over the field of meromorphic functions in $(x, \boldsymbol{u})$.

The paper is organized as follows. Some preliminary results and concepts regarding the geometric

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framework proposed for the study of nonlinear discrete-time dynamics are briefly recalled in Section 2. The main result is stated in Section 3. It is also noted that there is a specific result when a particular discrete-time dynamics is considered and shown how the extension of the results to general nonlinear continuous-time dynamics can be done. Finally, an academic example illustrates the computational aspects.

2. Recalls and preliminaries

The present recalls are issued from [11, 12].

Consider the nonlinear discrete-time dynamics

$$x_{k+1} = F(x_k, \boldsymbol{u}_k), \tag{1}$$

where $x \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, $F(\cdot, \boldsymbol{u}) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is analytic with $\text{rank}(\partial F/\partial \boldsymbol{u}) = m$, and $(x_e, \boldsymbol{u}_e) = (0, \boldsymbol{0})$ is an equilibrium pair and assume that Eq. (1) are drift invertible, i.e., $\boldsymbol{H}$: $F_0(x) := F(x, \boldsymbol{0})$ has full rank.

Let us note that the weaker submersivity assumption, [5] i.e.,

$$\text{rank}\left[\frac{\partial F(\cdot, \boldsymbol{u})}{\partial x} \vdots \frac{\partial F(\cdot, \boldsymbol{u})}{\partial \boldsymbol{u}}\right] = n$$

over the field of meromorphic functions in $(x, \boldsymbol{u})$ implies the existence of a regular static state feedback which renders the closed-loop system drift invertible. In our context, since the largest linearizable subsystem is feedback invariant, $\boldsymbol{H}$ can be set without any loss of generality.

Under $\boldsymbol{H}$, for $\boldsymbol{u}$ in a neighborhood $\mathscr{U}_0$ of $\boldsymbol{0}$, $F(x, \boldsymbol{u})$ is still invertible, and $F^{-1}(x, \boldsymbol{u})$ can be computed from $F_0^{-1}(x)$ according to its expansion with respect to $\boldsymbol{u}$. This ensures the existence and unicity of m functions, ${}_1G^0(x, \boldsymbol{u}), \ldots, {}_mG^0(x, \boldsymbol{u})$, parametrized by the control variables satisfying the set of partial differential equations

$$\frac{\partial F(\cdot, \boldsymbol{u})}{\partial u^{i_1}} = {}_{i_1}G^0(F(\cdot, \boldsymbol{u}), \boldsymbol{u}), \quad i_1 = 1, \ldots, m. \tag{2}$$

These functions can be computed, for $i_1 = 1, \ldots, m$, as

$$\begin{aligned} &{}_{i_1}G^0(x, \boldsymbol{u}) \\ &\quad \doteq \left(\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} F(\cdot, u^1, \ldots, u^{i_1} + \varepsilon, \ldots, u^m) \right)\bigg|_{F^{-1}(x, \boldsymbol{u})}, \end{aligned}$$

where u^i denotes the ith component of the control vector. By construction, ${}_{i_1}G^0(x, \boldsymbol{u})$ is an analytic vector field which admits the series expansion

$${}_{i_1}G^0(x, \boldsymbol{u}) = {}_{i_1}G_1^0(x) + \sum_{i_2=1}^{m} u^{i_2}\, {}_{i_1 i_2}G_2^0(x) + \cdots, \tag{3}$$

thus defining the canonical vector fields

$${}_{i_1}G_1^0(x) \doteq {}_{i_1}G^0(x, \boldsymbol{0}), \tag{4}$$

$${}_{i_1 i_2 \cdots i_s}G_s^0(x) \doteq \frac{\partial^{s-1}}{\partial u^{i_2} \cdots \partial u^{i_s}}\bigg|_{\boldsymbol{u}=\boldsymbol{0}} {}_{i_1}G^0(x, \boldsymbol{u})$$

$$\forall s > 1, \quad i_j = 1, \ldots, m.$$

In the sequel, for any $s \geqslant 1$, ${}_\eta G_s^0$ will stand for a generic vector field (4) with $\eta = i_1 \cdots i_s$. Recalling that the transport of any vector field $\theta(x)$ on $\mathbb{R}^n$ along F_0, is the vector field $\Gamma(x)$ satisfying $\Gamma(F_0(x)) = (\partial F_0(x)/\partial x) \times \theta(x)$, under $\boldsymbol{H}$, $\Gamma(x)$ is given by

$$\Gamma(x) := Ad_{F_0(x)}(\theta) := \left(\frac{\partial F_0(x)}{\partial x} \times \theta \right)\bigg|_{F_0^{-1}(x)}.$$

Setting $Ad^0_{F_0(x)}(\theta) := \theta$ and noting that for $p \geqslant 1$, $Ad_{F_0^p}(\theta) := Ad_{F_0} \circ Ad_{F_0^{p-1}}(\theta)$ where $F_0^p = F_0 \circ \cdots \circ F_0$ (p times), one iteratively introduces the transport of any ${}_\eta G_s^0$ along F_0^p as

$$\begin{aligned} {}_\eta G_s^p(x) &\doteq Ad^p_{F_0(x)}({}_\eta G_s^0(\cdot)) \\ &= \left(\frac{\partial F_0^p(x)}{\partial x} \times {}_\eta G_s^0(\cdot) \right)\bigg|_{F_0^{-p}(x)}. \end{aligned} \tag{5}$$

Moreover, it is immediately verified that these canonical vector fields are transformed, under any coordinates change $z = \Phi(x)$, into

$${}_\eta \tilde{G}_s^p(z) = \left(\frac{\partial \Phi(x)}{\partial x} \times {}_\eta G_s^p(x) \right)\bigg|_{\Phi^{-1}(z)} \quad \forall s \geqslant 1. \tag{6}$$

One of the advantages of the introduced tools, lies in the ability to express every coefficient of the expansion with respect to $\boldsymbol{u}$ of any composition of functions involving the dynamics as Lie derivatives with respect to the ${}_\eta G_s^0$. As an example, by considering an analytic function $\varphi^j(x) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, one computes

$$\frac{\partial \varphi^j \circ F(x, \boldsymbol{u})}{\partial u^i}\bigg|_{\boldsymbol{u}=\boldsymbol{0}} = L_{{}_iG_1^0(\cdot)}(\varphi^j)(x)\Big|_{x=F_0(x)},$$

$$\frac{\partial \varphi^j \circ F_0^p \circ F(x, \boldsymbol{u})}{\partial u^i}\bigg|_{\boldsymbol{u}=\boldsymbol{0}} = L_{{}_iG_1^p(\cdot)}(\varphi^j)(x)\Big|_{x=F_0^p(x)},$$

where $L_{{}_iG_1^j(\cdot)}$ indicates, as usual, the Lie derivative operator.

The notion of vector-relative degree can be thus characterized independently of the control variables, as described hereunder.

Definition 1. Let $\varphi(x)$ be an m-dimensional vector of analytic functions $\varphi^j(x)$, $j=1,\dots,m$. The discrete-time dynamics (1) has a strong vector-relative degree $\boldsymbol{r}=\{r_1,\dots,r_m\}$ with respect to $\varphi(x)$ at an equilibrium point x_e if and only if

(i) $L_{{}_\eta G_s^p}(\varphi^j(x))|_x=0$, $\forall s\geqslant 1$, $\forall \eta$, $p=0,\dots,r_j-2$,

(ii) there exists an index $i\in[1,m]$ such that $L_{{}_iG_1^{r_j-1}}(\varphi^j(x))|_{x_e}\neq 0$ and

$$\operatorname{rank}\left.\begin{pmatrix} L_{{}_1G_1^{r_1-1}}(\varphi^1)(x) & \cdots & L_{{}_mG_1^{r_1-1}}(\varphi^1)(x)\\ \vdots & \ddots & \vdots\\ L_{{}_1G_1^{r_m-1}}(\varphi^m(x) & \cdots & L_{{}_mG_1^{r_m-1}}(\varphi^m)(x)\end{pmatrix}\right|_{x_e}=m.$$

As proved by several authors (see for example [10]), if Eq. (1) has a strong vector-relative degree $\boldsymbol{r}$ with respect to $\varphi(x)$ then there exist a regular static state feedback and a coordinates change transforming the dynamics (1) into

$$z_{k+1}=\begin{pmatrix}A_1 & & \\ & \ddots & \\ & & A_s\end{pmatrix}z_k+\begin{pmatrix}B_1 & & \\ & \ddots & \\ & & B_s\end{pmatrix}\boldsymbol{v}_k,$$

$$\zeta_{k+1}=\chi(z_k,\zeta_k,\boldsymbol{v}_k), \tag{7}$$

where A_j and B_j are in the canonical Brunowskij form and are of dimension $r_j\times r_j$ and $r_j\times 1$, respectively. In this case the dynamics (1) is partially feedback linearizable with controllability indices $r_1,\dots,r_m$.

Let us here introduce some notations. Given any distribution Δ, $\bar{\Delta}$ denotes its involutive closure, and $\Delta^\perp$ denotes the associated codistribution. Finally, we denote by $\boldsymbol{G_1^i}:=[{}_1G_1^i\cdots{}_mG_1^i]$, $i\geqslant 0$ and by ${}_{i\eta}G_s^j$ the vector field ${}_{i_1i_2\dots i_s}G_s^j$ where $i_1=i$.

3. The largest linearizable subsystem

Problem statement. Given the dynamics (1) find the largest linearizable subsystem under regular static state feedback or, equivalently, fine $\varphi(x)$ with respect to which the dynamics (1) has a strong vector-relative degree $\boldsymbol{r}$ with $\sum_{i=1}^m r_i$ maximal.

This is discussed hereafter in the geometric framework following the lines of the continuous-time case. Suitable distributions are defined in terms of the canonical family of vector fields (4) and (5). The results obtained based on series expansions and local coordinates transformation have a local nature.

Since the expansion with respect to $\boldsymbol{u}$ of ${}_{i_1}G^0(\cdot,\boldsymbol{u})$ given in Eq. (3) leads to the introduction of an infinite family of vector fields, one needs to make a distinction between the vector fields of first order ${}_iG_1^0$, which represent the linear contribution of the control, and those of greater order ${}_{i_1\cdots i_s}G_s^0$ representing the contributions of the successive powers of the controls at the same time instant. Therefore, let us consider the set of distributions

$$\mathscr{Q}^0=\operatorname{span}\{{}_1G_1^0,\dots,{}_mG_1^0\},$$

$$\vdots$$

$$\mathscr{Q}^i=\operatorname{span}\{\bar{\mathscr{P}}^{i-1},{}_1G_1^i,\dots,{}_mG_1^i\},$$

$$\mathscr{P}^0=\operatorname{span}\{{}_{1\eta}G_s^0,\dots,{}_{m\eta}G_s^0\ s\geqslant 1\},$$

$$\vdots$$

$$\mathscr{P}^i=\operatorname{span}\{\mathscr{P}^{i-1},{}_{1\eta}G_s^i,\dots,{}_{m\eta}G_s^i\ s\geqslant 1\}.$$

which are supposed to have a constant dimension around x_e.

Remark. In order to compute $\mathscr{P}^0$ one only has to test the independency of at most n vector fields. Iteratively, $\mathscr{P}^i$ is generated by $\operatorname{span}\{\mathscr{P}^{i-1},Ad_{F_0}^i(\mathscr{P}^0)\}$.

The following results hold.

Proposition 1. *Let $h(x)$ be any vector field such that $\mathrm{d}h(x)\in(\bar{\mathscr{P}}^i)^\perp$, for $i\geqslant 1$, then $\mathrm{d}(h\circ F_0(x))\in(\bar{\mathscr{P}}^{i-1})^\perp$.*

Proof. The result follows by noting that

$$\frac{\partial(h\circ F_0(x))}{\partial x}\times\bar{\mathscr{P}}^{i-1}=\left.\frac{\partial h(x)}{\partial x}\right|_{F_0}\times\frac{\partial F_0}{\partial x}\times\bar{\mathscr{P}}^{i-1}=\left.\left(\frac{\partial h(x)}{\partial x}\times Ad_{F_0}(\bar{\mathscr{P}}^{i-1})\right)\right|_{F_0}$$

and that $Ad_{F_0}\bar{\mathscr{P}}^{i-1}\in\bar{\mathscr{P}}^i$ and $\mathrm{d}h(x)\in(\bar{\mathscr{P}}^i)^\perp$. □

Proposition 2. *Let $s_0=\dim\mathscr{Q}^0=m$ and $s_i=\dim\mathscr{Q}^i-\dim\bar{\mathscr{P}}^{i-1}$, for $i\geqslant 1$. The set of integers $\{s_0,s_1,\dots\}$ is nonincreasing.*

Proof. Suppose that $s_i > 0$, then there exist s_i vector fields $\lambda_j(x)$ such that $\lambda_j(x) \notin \bar{\mathscr{P}}^{i-1}$ and $\lambda_j(x) \in \mathscr{Q}^i$. Since $\mathscr{Q}^i = \text{span}\{\bar{\mathscr{P}}^{i-1}, {}_1G_1^i, \dots, {}_mG_1^i\}$, necessarily $\lambda_j(x) \in \text{span}\{\boldsymbol{G}_1^i\}$. Therefore, there exist s_i functions $\varphi_j(x) \in \text{span}\{\boldsymbol{G}_1^0\}$ such that $Ad_{F_0}^i \varphi_j(x) = \lambda_j(x)$. Moreover, $Ad_{F_0}^s \varphi_j(x) \in \text{span}\{\boldsymbol{G}_1^s\}$ and $\bar{\mathscr{P}}^{i-1} \cap \text{span}\{Ad_{F_0}^i \varphi_1(x) \cdots Ad_{F_0}^i \varphi_{s_i}(x)\} = \{0\}$.

It is now sufficient to show that

$$\bar{\mathscr{P}}^{i-2} \cap \text{span}\{Ad_{F_0}^{i-1} \varphi_1(x) \cdots Ad_{F_0}^{i-1} \varphi_{s_i}(x)\} = \{0\}.$$

In order to prove that $s_{i-1} \geqslant s_i$, let $\rho = \dim \bar{\mathscr{P}}^{i-1}$ then, there exist $n - \rho$ functions $h_j(x)$ such that $(\bar{\mathscr{P}}^{i-1})^{\perp} = \text{span}\{\text{d}h_1(x) \cdots \text{d}h_\rho(x)\}$. Therefore, for any set of nonzero coefficients $\alpha_1(x) \cdots \alpha_{s_i}(x)$ one finds

$$\frac{\partial h_r(x)}{\partial x} \times \sum_{j=1}^{s_i} \alpha_j(x) \times Ad_{F_0}^i \varphi_j(x) \neq 0, \quad r = 1, \dots, \rho. \tag{8}$$

Suppose that $\bar{\mathscr{P}}^{i-2} \cap \text{span}\{Ad_{F_0}^{i-1} \varphi_1(x) \cdots Ad_{F_0}^{i-1} \varphi_{s_i}(x)\} \neq \{0\}$. From Proposition 1, $\text{d}(h \circ F_0) \in (\bar{\mathscr{P}}^{i-2})^{\perp}$, there should exist some nonzero coefficients $\bar{\alpha}_1(x) \cdots \bar{\alpha}_{s_i}(x)$ such that for $r = 1, \dots, \rho$,

$$\frac{\partial(h_r \circ F_0)}{\partial x} \times \sum_{j=1}^{s_i} \bar{\alpha}_j(x) \times Ad_{F_0}^{i-1} \varphi_j(x) = 0,$$

i.e.

$$\frac{\partial h_r(x)}{\partial x} \times \sum_{j=1}^{s_i} \bar{\alpha}_j(F_0^{-1}) \times Ad_{F_0}^i \varphi_j(x) = 0,$$

which contradicts Eq. (8) thus proving that $s_{i-1} \geqslant s_i$. □

Denoting by $k_i^* = \text{card}\{s_j \geqslant i, j \geqslant 0\}$, it is straightforward to verify that $k_1^* \geqslant k_2^* \geqslant \cdots \geqslant k_m^*$ and $s_i = \text{card}\{k_j^* \leqslant i+1,\ 1 \leqslant j \leqslant m\}$ so that the next result can be stated.

Theorem 1. *The dynamics* (1) *is partially feedback linearizable with controllability indices* $k_1^*, k_2^*, \dots, k_m^*$.

Proof. Let $\bar{k}_1 > \bar{k}_2 > \cdots > \bar{k}_s$ be the distinct values of $(k_1^* \cdots k_m^*)$ and denote by p_i the multiplicity of $\bar{k}_i$. Clearly, $\sum_{i=1}^s p_i = m$. From the definition of k_i^*, it follows that $\bar{k}_1 = k_1^*$, $s_{\bar{k}_1 - 1} = p_1$ and $s_{\bar{k}_1} = 0$. Thus, there exists a p_1-dimensional column vector $\varphi_1(x)$ such that

$$\frac{\partial \varphi_1}{\partial x} \times \bar{\mathscr{P}}^{\bar{k}_1 - 2} = 0 \quad \text{and} \quad \text{rank}\left(\frac{\partial \varphi_1}{\partial x} \times \boldsymbol{G}_1^{\bar{k}_1 - 1}\right) = p_1.$$

It is easy to see that this implies

$$\frac{\partial(\varphi_1 \circ F_0^j)}{\partial x} \times \bar{\mathscr{P}}^{\bar{k}_1 - 2 - j} = 0, \quad j = 0, \dots, \bar{k}_1 - 2, \tag{9}$$

$$\text{rank}\left(\frac{\partial(\varphi_1 \circ F_0^{\bar{k}_1 - \bar{k}_l})}{\partial x} \times \boldsymbol{G}_1^{\bar{k}_1 - 1}\right) = p_1, \quad l = 0, \dots, s. \tag{10}$$

Iteratively one gets an m vector $(\varphi_1^{\mathrm{T}}(x), \dots, \varphi_s^{\mathrm{T}}(x))^{\mathrm{T}}$ satisfying

(i) $\varphi_j \circ F^i(x, \boldsymbol{u}) = \varphi_j \circ F_0^i(x)$, $i = 0, \dots, \bar{k}_j - 1$, which implies $(\partial \varphi_j / \partial x)\, \bar{\mathscr{P}}^i = 0$, $i = 0, \dots, \bar{k}_j - 2$ and

(ii)

$$\text{rank}\left.\begin{pmatrix} \dfrac{\partial \varphi_1}{\partial x} \times \boldsymbol{G}_1^{\bar{k}_1 - 1} \\ \vdots \\ \dfrac{\partial \varphi s}{\partial x} \times \boldsymbol{G}_1^{\bar{k}_s - 1} \end{pmatrix}\right|_{x = x_e} = p_1 + \cdots + p_s = m.$$

Condition (ii) implies the existence of a linearizing static state feedback and of a coordinates change $(z^{\mathrm{T}},\ \zeta^{\mathrm{T}})^{\mathrm{T}} := (\varphi(x)^{\mathrm{T}},\ \eta(x)^{\mathrm{T}})^{\mathrm{T}}$, (with $\varphi = (\varphi_j^{\mathrm{T}}, \dots, (\varphi_j \circ F_0^{\bar{k}_j - 1})^{\mathrm{T}}, j = 1, \dots, s)^{\mathrm{T}}$ and $\eta(x)$ appropriately chosen) transforming Eq. (1) into Eq. (7). The matrices A_j and B_j have dimensions $\bar{k}_j \times \bar{k}_j$ and $\bar{k}_j \times 1$ so that the linear subsystem has dimension $\sum_{i=1}^s p_i \bar{k}_i = \sum_{i=1}^m k_i^*$. □

The final step is to prove the maximality.

Lemma 1. *If the dynamics* (1) *is partially feedback linearizable with controllability indices* $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m$, *then* $k_i \leqslant k_i^*$.

Proof. Suppose that $k_i \leqslant k_i^*$ is not verified for all $i \in [1, m]$ and let j be the smallest index for which $k_j > k_j^*$. Then, there exist j independent functions $\phi_i \circ F_0^{k_i - k_j}$, for $i = 1, \dots, j$ such that

$$\frac{\partial(\phi_i \circ F_0^{k_i - k_j})}{\partial x} \times \bar{\mathscr{P}}^{k_j - 2} = 0$$

and

$$\text{rank}\left(\frac{\partial(\phi_i \circ F_0^{k_i - k_j})}{\partial x} \times \boldsymbol{G}_1^{k_j - 1}\right) = j$$

which implies that $s_{k_j - 1} = \dim \mathscr{Q}^{k_j - 1} - \dim \bar{\mathscr{P}}^{k_j - 2} = j$. On the other hand, since $k_j > k_j^*$, it follows from

Proposition 2 that $s_{k_j^*-1} \geqslant s_{k_j-1}$. One obtains $k_j^* = \text{card}\{s_i \geqslant j\} \geqslant k_j$, which contradicts the hypothesis. □

Theorem 1 and Lemma 1 enable us to state the main result.

Theorem 2. *The dynamics* (1) *is maximally feedback linearizable with controllability indices* $k_1^*, \ldots, k_m^*$.

A particular case. Let us consider the assumption

$$\boldsymbol{A} : {}_{\eta}G_s^0 \in \text{span}\{{}_1G_1^0 \cdots {}_mG_1^0\} \quad \forall s \geqslant 2,$$

which is equivalent to assuming

$$G^0(\cdot, \boldsymbol{u}) = [{}_1G_1^0 \cdots {}_mG_1^0]\alpha(x, \boldsymbol{u}),$$

where $\alpha(x, \boldsymbol{u})$ is an $m \times m$ matrix whose components are analytic functions with $\alpha(x, \boldsymbol{0})$ the identity matrix.

As shown in [2, 3], under $\boldsymbol{A}$, the distribution $\mathcal{Q}^0$ is involutive, and thus $\mathcal{Q}^0 \equiv \mathcal{P}^0 \equiv \bar{\mathcal{P}}^0$. Because of this one finds out iteratively that $\bar{\mathcal{P}}^i \equiv \bar{\mathcal{Q}}^i$ and thus $\mathcal{Q}^i = \text{span}\{\bar{\mathcal{Q}}^{i-1}, \boldsymbol{G_1^i}\}$. It follows that, to compute the largest linearizable subsystem, it is sufficient to consider $\mathcal{Q}^i$ with $s_i = \mathcal{Q}^i - \bar{\mathcal{Q}}^{i-1}$.

It is interesting to note that $\boldsymbol{A}$ is maintained under regular static state feedback and is necessary for achieving linear feedback equivalence (i.e. $\sum_{i=1}^m k_i^* = n$). In fact, generalizing the single-input case discussed in [11], one obtains the following result which illustrates the claimed analogy with the affine continuous-time case.

Theorem 3. *The dynamics* (1) *is feedback equivalent to a linear and controllable dynamics if and only if*

(i) $\mathcal{Q}^i \equiv \bar{\mathcal{P}}^i$ *for* $i = 0, \ldots, n-1$,

(ii) $\dim \mathcal{Q}^{n-1} = n$.

or equivalently if and only if

(i′) ${}_{\eta}G_s^0 \in \text{span}\{\boldsymbol{G_1^0}\}$ *for* $s \geqslant 1$,

(ii′) $\mathcal{Q}^i = \{\boldsymbol{G_1^0} \cdots \boldsymbol{G_1^i}\}$ *involutive and of constant rank around* x_e, *for* $i = 0, \ldots, n-2$,

(iii′) $\dim \mathcal{Q}^{n-1} = \dim\{\boldsymbol{G_1^0} \cdots \boldsymbol{G_1^{n-1}}\} = n$.

The general continuous-time case. Let us consider the continuous-time dynamics

$$\dot{x} = f_0(x(t)) + \sum_{s \geqslant 0} \frac{u^{i_1}(t) \cdots u^{i_s}(t)}{s!} {}_{i_1 \cdots i_s}g_s^0(x(t)),$$
$$i_j = 1, \ldots, m, \tag{11}$$

where $f_0(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and ${}_{i_1 \cdots i_s}g_s^0(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are analytic functions.

As already noted in [11], the geometric framework allows us to study the dynamics (11), which is nonlinear with respect to the control variables, through the extension of the results set in discrete time. To do so, denoting the usual Lie bracket $ad_{f_0}^p({}_{i\eta}g_s^0(\cdot)) := [f_0, ad_{f_0}^{p-1}({}_{i\eta}g_s^0(\cdot))]$, by ${}_{i\eta}g_s^p$ one considers the distributions

$$\mathcal{Q}^0 = \text{span}\{{}_1g_1^0, \ldots, {}_mg_1^0\},$$

$$\vdots$$

$$\mathcal{Q}^i = \text{span}\{\bar{\mathcal{P}}^{i-1}, {}_1g_1^i, \ldots, {}_mg_1^i\},$$

$$\mathcal{P}^0 = \text{span}\{{}_{1\eta}g_s^0, \ldots, {}_{m\eta}g_s^0 s \geqslant 1\},$$

$$\vdots$$

$$\mathcal{P}^i = \text{span}\{\mathcal{P}^{i-1}, {}_{1\eta}g_s^i, \ldots, {}_{m\eta}g_s^i s \geqslant 1\}$$

thus obtaining the following results:

Theorem 4. *The dynamics* (11) *is maximally feedback linearizable with controllability indices* $(k_1^*, \ldots, k_m^*)$.

Theorem 5. *The dynamics* (11) *is feedback equivalent to a linear and controllable system if and only if*

(i) $\mathcal{Q}^i \equiv \bar{\mathcal{P}}^i$ *for* $i = 0, \ldots, n-2$,

(ii) $\dim \mathcal{Q}^{n-1} = n$.

or equivalently if and only if

(i′) ${}_{\eta}g_s^0 \in \text{span}\{\boldsymbol{g_1^0}\}$ *for* $s \geqslant 1$,

(ii′) $\mathcal{Q}^i = \{\boldsymbol{g_1^0} \cdots \boldsymbol{g_1^i}\}$ *involutive and of constant rank around* x_e *for* $i = 1, \ldots, n-2$,

(iii′) $\dim \mathcal{Q}^{n-1} = \dim\{\boldsymbol{g_1^0} \cdots \boldsymbol{g_1^{n-1}}\} = n$.

Remark. The previous arguments have been developed with reference to drift invertible dynamics or to submersive dynamics transformed under a preliminary feedback into drift invertible ones. However, these results can be applied to any dynamics when assuming the existence of m suitable analytic parametrized functions ${}_iG^0(\cdot, \boldsymbol{u})$ satisfying the partial differential equations (2) together with the existence of their transport along the dynamics $F(\cdot, \boldsymbol{u})$ (executed at most $n-1$ times). This is illustrated by the following example specifically issued from [1] to reinforce the duality of the two approaches.

Example. Consider the discrete-time dynamics

$$x_1(k+1) = x_2(k) + u_1(k),$$

$x_2(k+1)=x_3(k)u_1(k),$

$x_3(k+1)=x_3(k)u_2(k),$

$x_4(k+1)=x_4(k)+u_1(k),$

which, even though not drift invertible satisfies conditions (2) with ${}_1G^0(\cdot,\boldsymbol{u})=(1\,x_3/u_2\,0\,1)^{\mathrm{T}}$, and ${}_2G^0(\cdot,\boldsymbol{u})=(0\,0\,x_3/u_2\,0)^{\mathrm{T}}$. Setting $(x_e,\boldsymbol{u}_e)=((0,0,a,0),(0,1))$ with $a\neq 0$, one easily computes

$${}_1G_1^0=(1\,x_3\,0\,1)^{\mathrm{T}},\qquad {}_{12}G_2^0=(0\;-x_3\,0\,0)^{\mathrm{T}},$$

$${}_2G_1^0=(0\,0\,x_3\,0)^{\mathrm{T}},$$

and verifies that $({}_{12i_3\cdots i_s}G_s^0,\ {}_{22\cdots i_s}G_s^0)\in\mathrm{span}\{{}_{12}G_2^0,\ {}_2G_1^0\}$, whereas ${}_{11i_3\cdots i_s}G_s^0={}_{21\cdots i_s}G_s^0=0$. In order to compute $\mathscr{Q}^1$ and $\mathscr{P}^1$ one considers the transport of ${}_1G_1^0, {}_2G_1^0,\ {}_{12}G_2^0$ along $F(\cdot,(0,1))$ given by

$${}_1G_1^1=(x_3\,0\,0\,1),\qquad {}_{12}G_2^1=(-x_3\,0\,0\,0),$$

$${}_2G_1^1=(0\,x_3\,0\,0).$$

It follows that

$$\mathscr{Q}^0=\left(\begin{pmatrix}1\\x_3\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\x_3\\0\end{pmatrix}\right),$$

$$\mathscr{P}^0=\left(\begin{pmatrix}1\\x_3\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\x_3\\0\end{pmatrix},\begin{pmatrix}0\\-x_3\\0\\0\end{pmatrix}\right)$$

which leads to $s_0=2$, and

$$\mathscr{Q}^1=\left(\begin{pmatrix}1\\x_3\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\x_3\\0\end{pmatrix},\begin{pmatrix}0\\-x_3\\0\\0\end{pmatrix},\begin{pmatrix}x_3\\0\\0\\1\end{pmatrix}\right).$$

For $a\neq 1$, $s_1\neq 1$ and $\dim\mathscr{Q}^1=4$ it results that $k_1^*=2$ and $k_2^*=1$. By considering $\mathscr{P}^{k_1^*-2}=\mathscr{P}^0$ one finds $\varphi_1(x)=x_4(k)-x_1(k)$ such that

$$\frac{\partial\varphi_1(x)}{\partial x}\times\mathscr{P}^0=0,$$

$$\mathrm{rank}\left[\frac{\partial\varphi_1(x)}{\partial x}\times({}_1G_1^1,\ {}_2G_1^1)\right]=1.$$

Under the coordinates change $(z_1,z_2,z_3,z_4)=(x_4-x_1,x_4-x_2,x_3,x_4)$ and the linearizing regular static state feedback law $u_1(k)=v_1(k)/(1-z_3(k))$, $u_2(k)=v_2(k)/z_3(k)$, one obtains the following dynamics:

$z_1(k+1)=z_2(k),$

$z_2(k+1)=z_4(k)+v_1(k),$

$z_3(k+1)=v_2(k),$

$$z_4(k+1)=z_4(k)+\frac{v_1(k)}{1-z_3(k)}$$

with controllability indices (2,1)

4. Conclusions

A method for computing the maximal partially feedback linearizable dynamics has been proposed in the discrete-time context. The developed arguments are dual to those proposed in [1]. The present approach, which does not require any accessibility assumption, is performed in a geometric framework which brings to control free results and provides a unifying methodology for discussing discrete-time, continuous-time and sampled dynamics as well as extending the results set for affine continuous-time systems to generically nonlinear continuous-time dynamics.

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