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The immersion under feedback of a multidimensional discrete-time non-linear system into a linear system

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This paper deals with the problem of finding conditions which ensure the reproducibility of the input–output behaviour of a linear analytic discrete-time system by means of a linear discrete-time system. On this basis the problem of modifying the system by feedback in order for it to enjoy such a property is posed and solved.

1. Introduction

Throughout this paper we shall deal with linear analytic discrete-time systems, i.e. discrete-time systems represented by input-state difference equations where the controls act linearly. Linear analytic continuous-time systems have been widely investigated in the last decade (Lobry 1970, Sussman and Jurdjevic 1972, Hermann and Krener 1977).

Some problems which are interesting from the control point of view have recently been solved by modifying a given system by means of feedback (Freund 1975, Sinha 1977, Brockett 1978, Jakubczyk and Respondek 1980, Hirschorn 1981, Claude 1982). The approach is essentially geometric and based on the tools of differential geometry. Non-linear discrete-time systems have been investigated mainly by following an algebraic approach (Sontag 1979). The approach developed here is mainly based on the notion of generating series (Fliess 1980, 1981) and some extensions on the input–output behaviour (Normand-Cyrot 1982). On this basis we give, in Theorem 1, necessary and sufficient conditions under which, for any given initial state, the input–output behaviour of the initialized system coincides with that of a linear initialized system; moreover we look for a linear system which reproduces all the input–output behaviour characteristics of the non-linear one.

The formal statement of this problem brings us to the concept of immersion recently introduced by Fliess and Kupka (1983). A preliminary study of immersion into a linear system of a multi-input multi-output non-linear discrete-time system, performed in § 3, is followed in § 4 by immersion under feedback. By the latter we mean the problem of finding a feedback control law under which the modified system is immersed into a linear one. Some links with other control problems, e.g. input–output decoupling, are briefly discussed.

The results presented here generalize to the multi-input multi-output situation some results presented in (Monaco and Normand-Cyrot 1982) for

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the one-input one-output case ; and they extend to the class of discrete-time systems, some results recently obtained by Claude *et al.* (1982) for continuous-time non-linear systems.

The following section deals with some preliminaries.

2. Input–output development and generating series

The notion of a generating series, which characterizes the input–output behaviour of a non-linear system, was introduced by Fliess (1980). For the sake of completeness, in the sequel there is briefly presented an extended notion of generating series associated with a non-linear discrete-time system given by analytic (or polynomial) difference equations ; for a more complete treatment, see Normand-Cyrot (1982).

Consider the *linear analytic discrete-time system*

$$x(t+1)=x(t)+f(x(t))+\sum_{i=1}^{p} u_i(t)g_i(x(t))$$

$$=x(t)+f(x(t))+\mathbf{g}(x(t))\mathbf{u}(t) \qquad (1)$$

(Σ)

$$y(t)=h(x(t))\ ; \quad x(t_0)=x_0\in\mathbb{R}^N \qquad (2)$$

where $x(t)\in\mathbb{R}^N$; $f, g_1, \ldots, g_p : \mathbb{R}^N\rightarrow\mathbb{R}^N$ and $h : \mathbb{R}^N\rightarrow\mathbb{R}$ are analytic functions on $\mathbb{R}^N$; the input $u=(u_1, \ldots, u_p)$ are real-valued.

The input–output behaviour of the initialized system Σ_{x_0}, can be represented by means of a generating series

$$\mathscr{G}=\sum_{\alpha\geqslant 0} y(\alpha)$$

The notion of a generating series can be given formally in the following manner. Let $\mathbb{R}[\![u_1, \ldots, u_p]\!]$ be the $\mathbb{R}$-algebra of commutative formal power series, and denote by

$$\mathbb{R}[\![u_1, \ldots, u_p]\!]^{\hat{\otimes}\alpha} \triangleq \mathbb{R}[\![u_1, \ldots, u_p]\!]\otimes\ldots\otimes\mathbb{R}[\![u_1, \ldots, u_p]\!]$$

the complete tensor product of α copies of $\mathbb{R}[\![u_1, \ldots, u_p]\!]$; each element of $\mathbb{R}[\![u_1, \ldots, u_p]\!]^{\hat{\otimes}\alpha}$ is a formal power series with $\alpha\cdot p$ variables

$$u_1(0), \ldots, u_p(0), \ldots, u_1(\alpha-1), \ldots, u_p(\alpha-1)$$

Let

$$\hat{T}(\mathbb{R}[\![u_1, \ldots, u_p]\!])^\wedge = \widehat{\bigoplus_{\alpha\geqslant 0}}\ \mathbb{R}[\![u_1, \ldots, u_p]\!]^{\hat{\otimes}\alpha}$$

be the *complete tensor algebra* ; an element of it can be represented by

$$\mathscr{G}=\sum_{\alpha\geqslant 0}\mathscr{G}_\alpha$$

where each $\mathscr{G}_\alpha$ belonging to $\mathbb{R}[\![u_1, \ldots, u_p]\!]^{\hat{\otimes}\alpha}$ is a convergent power series.

Then $\mathscr{G}$ is a *generating series* for an input–output system described by

$$\begin{aligned}
&\text{at } t=0, \quad y(0)=\mathscr{G}_0 \in \mathbb{R}\\
&\text{at } t=1, \quad y(1)=\mathscr{G}_1(u_1(0), \ldots, u_p(0)) \in \mathbb{R}[\![u_1, \ldots, u_p]\!]\\
&\vdots\\
&\text{at } t=\alpha, \quad y(\alpha)=\mathscr{G}_\alpha(u_1(0), \ldots, u_p(0), \ldots, u_1(\alpha-1), \ldots,\\
&\qquad\qquad\qquad u_p(\alpha-1)) \in \mathbb{R}[\![u_1, \ldots, u_p]\!]^{\hat{\otimes}\alpha}
\end{aligned}$$

The input–output behaviour of Σ_{x_0} can be represented by means of $\mathscr{G}$, an element of $\hat{T}(\mathbb{R}[\![u_1, \ldots, u_p]\!])^\wedge$, as

$$\mathscr{G}=\sum_{\alpha\geqslant 0} y(\alpha)=h(x_0)+h(x_0+f(x_0)+\mathbf{g}(x_0)\mathbf{u}(0))+\ldots \tag{3}$$

Moreover $\mathscr{G}$ can be developed as a series in the commutative variables $u_1(0), \ldots, u_p(0), u_1(1), \ldots, u_p(1), \ldots$.

Remark 1

If the output $\mathbf{h}=(h_1, \ldots, h_q)$ is q-dimensional, we construct a vector of generating series $\mathscr{G}=(\mathscr{G}^1, \ldots, \mathscr{G}^q)$, where $\mathscr{G}^i$ is the generating series corresponding to the ith component h_i of the output $\mathbf{h}$.

Example 1

Let us consider the linear system

$$\begin{aligned}
x(t+1)&=Ax(t)+\sum_{i=1}^{p} u_i(t)B_i\\
y(t)&=Cx(t)
\end{aligned}$$

where $x(t)\in\mathbb{R}^N$, x_0 is the initial state and A_1, $B_1, \ldots, B_p$ and C are matrices of appropriate dimensions.

The output at time t is given by

$$y(t)=CA^t x_0+\sum_{k=0}^{t-1}\sum_{i=1}^{p} CA^{t-1-k}B_i u_i(k)$$

It can be rewritten as an element of $\mathbb{R}[\![u_1, \ldots, u_p]\!]^{\otimes\alpha}$ be setting $u_0(\tau)=1$ for all $\tau\geqslant 0$; one has

$$y(t)=CA^t x_0 {u_0}^{\otimes t}+\sum_{k=0}^{t-1}\sum_{i=1}^{p} CA^{t-1-k}B_i {u_0}^{\otimes t-1-k}\otimes u_i\otimes {u_0}^{\otimes k}$$

The generating series is then

$$\mathscr{G}=\sum_{\alpha\geqslant 0} y(\alpha)=C(I-Au_0)^{\otimes -1}\left(x_0+\sum_{i=1}^{p} B_i u_i\otimes {u_0}^{\otimes \alpha}\right)$$

where $(I-Au_0)^{\otimes -1}=I+A{u_0}^{\otimes 1}+A^2{u_0}^{\otimes 2}+\ldots+A^m{u_0}^{\otimes m}+\ldots$

Example 2

Let us consider the homogeneous bilinear system

$$x(t+1)=\left(A_0+\sum_{i=1}^{p}u_i(t)A_i\right)x(t)$$

$$y(t)=Cx(t)$$

where $x(t)\in\mathbb{R}^N$, x_0 is given, and $A_0, A_1, \ldots, A_p$ and C are matrices of appropriate dimensions.

It can easily be verified that the generating series is given by

$$\mathscr{G}=\sum_{\alpha\geqslant 0}y(\alpha)=C\left(I+\sum_{\nu\geqslant 0}\sum_{i_\nu,\ldots,i_0=0}^{p}A_{i_\nu}\ldots A_{i_0}u_{i_\nu}(\nu)\ldots u_{i_0}(0)\right)x_0$$

It can be rewritten as

$$\mathscr{G}=C\left(I+\sum_{\nu\geqslant 0}\sum_{i_\nu,\ldots,i_0=0}^{p}A_{i_\nu}\ldots A_{i_0}u_{i_\nu}\otimes\ldots\otimes u_{i_0}\right)x_0$$

$$\triangleq C\left(I-\sum_{i=0}^{p}A_iu_i\right)^{\otimes -1}x_0$$

A more useful expression of the generating series associated with Σ_{x_0} can be obtained by means of series development of the composition of analytic functions.

Let us introduce the following notations. Let $f, g:\mathbb{R}^N\to\mathbb{R}^N$ be analytic functions on $\mathbb{R}^N$; L_f will denote the directional derivative operator associated with the function f defined by

$$L_f\triangleq\sum_{i=1}^{N}f_i\frac{\partial}{\partial x_i}\quad(f_i=i\text{th component of }f)$$

$L_f\otimes L_g$ will denote the tensor product of directional derivative operators defined by

$$L_f\otimes L_g\triangleq\sum_{i,j=1}^{N}f_ig_j\frac{\partial^2}{\partial x_i\,\partial x_j}$$

Remark 2

The tensor product just defined is commutative and associates with two differential operators of the first order a second-order differential operator.

Let us now define the following operator

$$\exp_{\otimes}L_f=\Delta_f\triangleq I+L_f+\frac{1}{2!}L_f^{\otimes 2}+\ldots+\frac{1}{p!}L_f^{\otimes p}+\ldots$$

where $L_f^{\otimes p}=L_f\otimes\ldots\otimes L_f$ (p times).

It follows from the theory of formal differential groups introduced by Ritt (1951) that, given an analytic function $h:\mathbb{R}^N\to\mathbb{R}$

$$h\circ(I+f)(x)=\Delta_f(h)\big|_x,\quad\forall x\in\mathbb{R}^N\tag{4}$$

where $|_x$ denotes the evaluation at x.

It is easy to verify, by the definition of Δ_f, that

$$\Delta_{f+g}(h)=\Delta_f\otimes\Delta_g(h) \tag{5}$$

From (4) and (5), for any $u\in\mathbb{R}$, we obtain the following

$$\Delta_{f+gu}(h)=\Delta_f(h)+\sum_{n\geqslant 1}\frac{u^n}{n!}\,\Delta_f\otimes L_g{}^{\otimes n}(h) \tag{6}$$

Let us now consider the initialized system Σ_{x_0} and suppose at first, for the sake of simplicity, that the input u be scalar. Direct computations lead to

$$\begin{aligned}
y(0)&=h\big|_{x_0}\\
y(1)&=h\circ(I+f+u(0)g)(x_0)=\Delta_{f+u(0)g}(h)\big|_{x_0}\\
y(2)&=h\circ(I+f+u(1)g)(x_1)=\Delta_{f+u(1)g}(h)\big|_{(I+f+u(0)g)(x_0)}\\
&\;\vdots\qquad\qquad\qquad\qquad\quad=\Delta_{f+u(0)g}\circ\Delta_{f+u(1)g}(h)\big|_{x_0}\\
y(t+1)&=\Delta_{f+u(0)g}\circ\ldots\circ\Delta_{f+u(t)g}(h)\big|_{x_0}
\end{aligned}$$

Hence the generating series (3) can be written as

$$\mathscr{G}=\sum_{t\geqslant 0}y(t)=h\big|_{x_0}+\sum_{t\geqslant 1}\Delta_{f+gu(0)}\circ\ldots\circ\Delta_{f+gu(t-1)}(h)\big|_{x_0} \tag{7}$$

By means of (6), the generating series (7) can be expressed as a series in powers of the control $u(\tau)$ at different instants of time $\tau\geqslant 0$. In this way we obtain a ‘ discrete Volterra extension ’ of $\mathscr{G}$, i.e.

$$\begin{aligned}
\mathscr{G}=\sum_{t\geqslant 0}\Bigg[w_0(t)+\sum_{\tau_1=0}^{t-1}w_1(t,\tau_1)u(\tau_1)+\sum_{\tau_2\geqslant\tau_1=0}^{t-1}w_2(t,\tau_2,\tau_1)u(\tau_2)u(\tau_1)+\ldots\\
+\sum_{\tau_s\geqslant\ldots\geqslant\tau_1=0}^{t-1}w_s(t,\tau_s,\ldots,\tau_1)u(\tau_s)\ldots u(\tau_1)+\ldots\Bigg]
\end{aligned} \tag{8}$$

where w_i is called the ith-order Volterra kernel of the extension (8). It is not difficult to verify that the Volterra kernels can be expressed by means of iterative applications of differential derivative operators and tensor products of them ; the computation leads to

$$\begin{aligned}
w_0(t)&=\Delta_f{}^t(h)\big|_{x_0}\\
w_1(t,\tau_1)&=\Delta_f{}^{\tau_1}\circ\Delta_f\otimes L_g\circ\Delta_f{}^{t-1-\tau_1}(h)\big|_{x_0}\\
w_2(t,\tau_2,\tau_1)&=\begin{cases}\Delta_f{}^{\tau_1}\circ\Delta_f\otimes L_g\circ\Delta_f{}^{\tau_2-\tau_1-1}\circ\Delta_f\otimes L_g\circ\Delta_f{}^{t-1-\tau_2}(h)\big|_{x_0}, & \tau_2>\tau_1\\[2ex]
\dfrac{1}{2!}\,\Delta_f{}^{\tau_1}\circ\Delta_f\otimes L_g{}^{\otimes 2}\circ\Delta_f{}^{t-1-\tau_1}(h)\big|_{x_0}, & \tau_2=\tau_1\end{cases}\\
&\;\vdots
\end{aligned}$$

In the sequel we shall restrict our attention to the first-order kernels; for this reason we specify the development of $y(t)$, $t \geqslant 0$, in powers of u up to the term of degree 1. We have

$$y(t)=\Delta_f{}^t(h)\big|_{x_0}+\sum_{\tau=0}^{t-1}\Delta_f{}^{\tau}\circ\Delta_f\otimes L_g\circ\Delta_f{}^{t-1-\tau}(h)\big|_{x_0}u(\tau) \\ +\text{terms of degree} \geqslant 2 \text{ in the control } u(\tau), \quad \tau \geqslant 0 \qquad (9)$$

Finally we note that when the input is p-dimensional, the same computation leads to

$$y(t)=\Delta_f{}^t(h)\big|_{x_0}+\sum_{i=1}^{p}\sum_{\tau=0}^{t-1}\Delta_f{}^{\tau}\cdot\Delta_f\otimes L_{g_i}\cdot\Delta_f{}^{t-1-\tau}(h)\big|_{x_0}u_i(\tau) \\ +\text{higher-order terms} \qquad (10)$$

3. Immersion of Σ into a linear system

First let us give the definition of immersion, a concept introduced recently by Fliess and Kupka (1983) with reference to a continuous-time non-linear differential system. It will be evident that the concept of immersion is limited to the capability of reproducing, by means of another system, the input–output behaviour of a given one.

In the sequel we will refer to Σ as the multi-input multi-output system defined by (1), and

$$y_j(t)=h_j(x(t)), \quad j=1, \dots, q \qquad (11)$$

where $h_j: \mathbb{R}^N \rightarrow \mathbb{R}$, $j=1, \dots, q$, are assumed to be analytic.

Let Σ' be the system given by

$$(\Sigma') \quad \begin{aligned} x'(t+1)&=x'(t)+f'(x'(t))+\sum_{i=1}^{p} g'_i(x'(t))u_i(t) && (12)\\ y'_j(t)&=h'_j(x'(t)), \quad j=1, \dots, q, \quad x'(t_0)=x_0{}'\in\mathbb{R}^{N'} && (13)\end{aligned}$$

where $x'(t)\rightarrow\mathbb{R}^{N'}$ and $f': \mathbb{R}^{N'}\rightarrow\mathbb{R}^{N'}$; $g'_i: \mathbb{R}^{N'}\rightarrow\mathbb{R}^{N'}$, $i=1, \dots, p$; $h'_j: \mathbb{R}^{N'}\rightarrow\mathbb{R}$, $j=1, \dots, q$ are analytic functions.

Definition 1

Σ is said to be immersed into Σ' if there exists an analytic map $\tau: \mathbb{R}^N\rightarrow\mathbb{R}^{N'}$ such that at $x\in\mathbb{R}^N$ and $\tau(x)=x'\in\mathbb{R}^{N'}$ the initialized systems Σ_x and $\Sigma'_{x'}$ have the same input–output behaviour; i.e. the two generating series at x and x' coincide.

Remark 3

Note that the map $\tau: \mathbb{R}^N\rightarrow\mathbb{R}^{N'}$ is not required to be 'onto'; this ensures a richer input–output behaviour of the system Σ' with respect to Σ. If τ is 'onto', then for any $x'\in\mathbb{R}^{N'}$ there exists at least an $x\in\mathbb{R}^N$ such that $\Sigma'_{x'}$ and Σ_x coincide. This property holds true if Σ' is controllable.

In the sequel we will give necessary and sufficient conditions for the system Σ to be immersed into a linear discrete-time system.

Let us denote by θ the vector space generated by the functions

$$h_1,\ \Delta_f h_1,\ \ldots,\ h_2,\ \Delta_f h_2,\ \ldots,\ h_q,\ \Delta_f h_q,\ \ldots$$

Moreover θ_i will denote the vector space generated by h_i, $\Delta_f h_i$, … .

Lemma 1

If θ has finite dimension, say k, then

(i) a basis of θ can be chosen as

$$\{h_1, \ldots, \Delta_f{}^{l_1-1}(h_1), h_2, \ldots, \Delta_f{}^{l_2-1}(h_2), \ldots, h_q, \ldots, \Delta_f{}^{l_q-1}(h_q)\}$$

where $l_1+l_2+\ldots+l_q=k$; $l_i \geqslant 0$

(ii) each θ_i has finite dimension, say k_i, $i=1, \ldots, q$, and

$$\max\{k_1, \ldots, k_q\} \leqslant k \leqslant k_1+\ldots+k_q$$

Proof

First consider the set of functions $h_1, \ldots, h_q$ and assume that the last $q-n_1$ ($n_1 \leqslant q$) can be expressed as linear combinations of the first n_1. Then $h_1, \ldots, h_{n_1}$ can be assumed to belong to a basis of θ ; consider now the functions $\Delta_f h_1, \ldots, \Delta_f h_{n_1}$ (note that for all $j>n_1$, $\Delta_f h_j$ can be expressed as a linear combination of these functions), and assume, by means of an eventual reordering, that the last n_1-n_2 ($n_2 \leqslant n_1$) functions are linearly dependent on $h_1, \ldots, h_{n_1}, \Delta_f h_1, \ldots, \Delta_f h_{n_2}$; those functions can be assumed to belong to the wanted basis of θ. Consider now the functions $\Delta_f{}^2\, h_1, \ldots, \Delta_f{}^2\, h_{n_2}$ and assume, by means of an eventual reordering, that the last n_2-n_3 ($n_3 \leqslant n_2$) are linearly dependent on $h_1, \ldots, h_{n_1}, \Delta_f h_1, \ldots, \Delta_f h_{n_2}, \Delta_f{}^2\, h_1, \ldots, \Delta_f{}^2\, h_{n_3}$; those functions can be assumed to belong to the wanted basis of θ. By iterating this procedure at least k times, (i) is proved.

(ii) can be easily proved starting from (i) and performing simple changes of bases. ■

Theorem 1

The system Σ is immersed into the linear system

$$\tilde{x}(t+1)=A\tilde{x}(t)+B\mathbf{u}(t) \tag{14}$$

(Σ_L)

$$y(t)=C\tilde{x}(t), \quad \tilde{x}(t_0)=\tilde{x}(0)=\tilde{x}_0 \tag{15}$$

where $\tilde{x}(t)\in\mathbb{R}^k$, $u(t)\in\mathbb{R}^p$, $y(t)\in\mathbb{R}^q$ and A, B and C are matrices of appropriate dimensions, if and only if

(i) dim $\theta=k$.

(ii) $$\Delta_f \otimes L_{g_i}(\gamma)\big|_x=\text{constant}, \quad \forall i\in\{1, \ldots, p\}, \quad \forall\gamma\in\theta, \quad \forall x\in\mathbb{R}^N$$

$$\Delta_f \otimes L_{g_{i_\nu}} \otimes \ldots \otimes L_{g_{i_1}}(\gamma)\big|_x=0, \quad \forall\nu\geqslant 2, \quad i_\nu, \ldots, i_1\in\{1, \ldots, p\}, \quad \forall\gamma\in\theta, \quad \forall x\in\mathbb{R}^N$$

Proof

(*a*) *Sufficiency*. Suppose that $\dim \theta = k$ and let us choose a basis for θ as in (i) of Lemma 1. Let us define the application $\tau : \mathbb{R}^N \to \mathbb{R}^k$ as

$$\tilde{x} = \tau(x) = \begin{bmatrix} (h_1)|_x \\ \vdots \\ \Delta_f^{l_1-1}(h_1)|_x \\ \vdots \\ (h_q)|_x \\ \vdots \\ \Delta_f^{l_q-1}(h_q)|_x \end{bmatrix} \tag{16}$$

Under the application τ, just defined, Σ is immersed into Σ_L with

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & \cdots & & & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & & & & \vdots \\ 0 & \cdots & & 0 & 1 & 0 & \cdots & & & \cdots & 0 \\ a_1^{\,1} & \cdots & & & a_{l_1}^{\,1} & \cdots & a^1_{k-l_q+1} & & & \cdots & a_k^{\,1} \\ \vdots & & & & \vdots & & \vdots & & & & \vdots \\ 0 & \cdots & & & 0 & 0 \; \dots \; 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots & & & & \vdots \\ 0 & \cdots & & & 0 & 0 \; \dots \; 0 & 0 & \cdots & & 0 & 1 \\ a_1^{\,q} & \cdots & & & a_{l_1}^{\,q} & \cdots & a^q_{k-l_q+1} & & & \cdots & a_k^{\,q} \end{bmatrix} \in \mathbb{R}^{k \times k} \tag{17}$$

$$\mathbf{b}_i = \begin{bmatrix} b_{1i} \\ \vdots \\ b_{l_1 i} \\ \vdots \\ b_{k-l_q+1,i} \\ \vdots \\ b_{ki} \end{bmatrix} \in \mathbb{R}^{k \times 1}, \quad i = 1, \dots, p$$

$$C = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & & \dots & 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & & \dots & 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{q \times k}$$

and

$$\Delta_f^{l_i} h_i = \sum_{j=1}^{l_1} a_j^{\,i} \, \Delta_f^{j-1}(h_1) + \sum_{j=1}^{l_2} a_{l_1+j}^{\,i} \, \Delta_f^{j-1}(h_2) + \dots$$
$$+ \sum_{j=1}^{l_q} a_{l_1+\dots+l_{q-1}+j}^{\,i} \, \Delta_f^{j-1}(h_q), \quad i = 1, \dots, q$$

$$b_{ji} = \Delta_f \otimes L_{g_i} \circ \Delta_f^{j-1}(h_1), \quad i \in \{1, \dots, p\}\,; \quad j \in \{1, \dots, l_1\}$$
$$\vdots$$
$$b_{ji} = \Delta_f \otimes L_{g_i} \circ \Delta_f^{j-1-(l_1+\dots+l_{q-1})}(h_q), \quad i \in \{1, \dots, p\}\,; \quad j \in \{k-l_q+1, \dots, k\}$$

For, note that, with $j \neq l_1, l_1+l_2, \ldots, l_1+l_2+\ldots+l_q$, and $j \in \{l_1+\ldots+l_{r-1}, \ldots, l_1+\ldots+l_r\}$, we have

$$\begin{aligned}\Delta_{f+gu}(\tilde{x}_j) &= \Delta_f \circ \Delta_f^{j-1}(h_r)\big|_x + \sum_{i=1}^{p} u_i \Delta_f \otimes L_{g_i} \circ \Delta_f^{j-1}(h_r)\big|_x \\ &= \tilde{x}_{j+1} + \sum_{i=1}^{p} u_i b_{ji}\end{aligned}$$

Moreover, when $j = l_1+\ldots+l_r$, $r = 1, \ldots, q$, we have

$$\begin{aligned}\Delta_{f+gu}(\tilde{x}_j) &= \Delta_f^{l_r}(h_r)\big|_x + \sum_{i=1}^{p} u_i \Delta_f \otimes L_{g_i} \circ \Delta_f^{l_1+l_2+\ldots+l_{r-1}}(h_r)\big|_x \\ &= \sum_{j=1}^{l_1} a_j^r \, \Delta_f^{j-1}(h_1)\big|_x + \ldots \\ &\qquad + \sum_{j=1}^{l_q} a_{l_1}{}^r{}_{+\ldots+l_{q-1}+j} \, \Delta_f^{j-1}(h_q)\big|_x + \sum_{i=1}^{p} u_i b_{l_1+\ldots+l_r, i}\end{aligned}$$

(*b*) *Necessity*. Suppose that Σ is immersed, by means of $\tau : \mathbb{R}^N \to \mathbb{R}^{N'}$, into the linear system Σ_L. At $x \in \mathbb{R}^N$ and $\tilde{x} = \tau(x) \in \mathbb{R}^{N'}$ the generating series of Σ and that of Σ_L, given by

$$\mathscr{G} = \sum_{l \geq 0} CA^l \tilde{x} + \sum_{\alpha \geq 0} \sum_{i=1}^{p} \sum_{l \geq 0} CA \mathbf{b}_i u_i(\alpha) \tag{18}$$

coincide by definition. Because $\sum_{l \geq 0} CA^l \tilde{x}$ corresponds to $\sum_{l \geq 0} \Delta_f^l(\mathbf{h})\big|_x$ in (9), the condition (i) holds.

Moreover, the $(q \times 1)$ vector $\sum_{l \geq 0} CA^l \mathbf{b}_i$ associated with $u_i(\alpha)$ corresponds to $\sum_{l \geq 0} \Delta_f^\alpha \circ \Delta_f \otimes L_{g_i} \Delta_f^l(\mathbf{h})\big|_x$ in (9). Hence for $\alpha = 0$ this last vector is constant; the terms in (9) corresponding to products of inputs of degree ≥ 2 are equal to zero, and the proof is complete. ■

4. Immersion by feedback into a linear system

The present section is devoted to the problem of modifing Σ by means of a feedback control law in order to let the modified system be immersed into a linear system. In order to solve the problem posed, we need to introduce the notion of 'index' associated with each output channel of system Σ. This notion is the natural extension to the discrete-time context of the 'characteristic number' introduced with reference to continuous-time differential systems (Hirschorn 1981, Isidori *et al.* 1981, Claude 1982, Claude *et al.* 1983).

Let us associate with each output channel y_i, $i = 1, \ldots, q$, the relative index d_i, defined as the smallest integer such that

(i) $\Delta_f \otimes L_{g_{i_\nu}} \otimes \ldots \otimes L_{g_{i_1}} \Delta_f^r(h_i)\big|_x = 0, \quad \forall x \quad \forall r \leq d_i,$
$$\forall \nu, \quad i_\nu, \ldots, i_1 \in \{1, \ldots, p\} \tag{19}$$

(ii) $\exists \bar{\nu} \in N^+$ and a sequence $\bar{i}_1, \ldots, \bar{i}_{\bar{\nu}}$ such that

$$\Delta_f \otimes L_{g_{\bar{i}_{\bar{\nu}}}} \otimes \ldots \otimes L_{g_{\bar{i}_1}} \Delta_f^{d_i}(h_i)\big|_x \neq 0, \quad \forall x \in \text{open and dense in } \mathbb{R}^N \tag{20}$$

Remark 4

It can be easily proved by means of (10) that $t = d_i + 1$ is the first instant of time at which the ith output is affected by at least one input acting at time $t_0 = 0$.

Remark 5

If $\dim \theta = k$, then $d_i \leqslant k_i - 1$, $i = 1, \ldots, q$. For, suppose that $d_i \geqslant k_i$ for some i; then

$$\Delta_f^{d_i} h_i = \sum_{j=1}^{k_i} c_j \Delta_f^{j-1} h_i$$

and

$$\begin{aligned} &\Delta_f \otimes L_{g_{i_\nu}} \otimes \ldots \otimes L_{g_{i_1}} \Delta_f^{d_i} h_i \\ &\quad = \sum_{j=1}^{k_i} c_j \Delta_f \otimes L_{g_{i_\nu}} \otimes \ldots \otimes L_{g_{i_1}} \Delta_f^{j-1} h_i = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \nu, \quad i_1, \ldots, i_\nu \in \{1, \ldots, p\} \end{aligned}$$

which is in contradiction with the definition of d_i.

Definition 2

Σ is said to be immersed by feedback into a linear system if there exist analytic functions α_i and β_{ij}, $i = 1, \ldots, p$, $j = 1, \ldots, p'$, such that the resulting feedback system, Σ_F with

$$\tilde{f} = f + \sum_{i=1}^{p} \alpha_i g_i, \quad \tilde{g}_l = \sum_{i=1}^{p} g_i \beta_{il}, \quad l = 1, \ldots, p' \tag{21}$$

is immersed into a linear system.

Lemma 2

(i) $$\Delta^r_{f + \Sigma \alpha_i g_i}(h_i) = \Delta_f^r(h_i), \quad 0 \leqslant r \leqslant d_i, \quad i = 1, \ldots, q \tag{22}$$

(ii) $$\begin{aligned} &\Delta_{f + \Sigma \alpha_i g_i} \otimes L_{(g\beta)_{i_\nu}} \otimes \ldots \otimes L_{(g\beta)_{i_1}} \Delta^r_{f + \Sigma \alpha_i g_i}(h_i) = 0, \quad i = 1, \ldots, q, \\ &\qquad\qquad 0 \leqslant r \leqslant d_i - 1; \quad \forall \nu, \forall i_1, \ldots, i_\nu \in \{1, \ldots, p\} \end{aligned} \tag{23}$$

Proof

(i) For $r = 0$ it is true. Suppose it is verified for $0 \leqslant r \leqslant d_i - 1$; we have

$$\begin{aligned} &\Delta^r_{f + \Sigma g_i \alpha_i} \Delta_f^r(h_i) \\ &\quad = \Delta_f^{r+1}(h_i) + \sum_{n \geqslant 1} \sum_{j_n, \ldots, j_1 = 1}^{p} \frac{\alpha_{j_n} \ldots \alpha_{j_1}}{n!} \Delta_f \otimes L_{g_{j_n}} \otimes \ldots \\ &\qquad\qquad \otimes L_{g_{j_1}} \Delta_f^r(h_i) + \text{ higher order terms} \\ &\quad = \Delta_f^{r+1}(h_i) \text{ (because of the definition of } d_i) \end{aligned}$$

(ii) $$\begin{aligned} &\Delta_{f + \Sigma \alpha_i g_i} \otimes L_{(g\beta)_{i_\nu}} \otimes \ldots \otimes L_{(g\beta)_{i_1}} \Delta^r_{f + \Sigma \alpha_i g_i}(h_i) \\ &\quad = \sum_{j_\nu, \ldots, j_1 = 1}^{p} \beta_{j_\nu i_\nu} \ldots \beta_{j_1 i_1} (\Delta_{f + \Sigma \alpha_i g_i} \otimes L_{g_{j_\nu}} \otimes \ldots \otimes L_{g_{j_1}} \Delta_f^r(h_i)) \\ &\quad = \sum_{j_\nu, \ldots, j_1 = 1}^{p} \beta_{j_\nu i_\nu} \ldots \beta_{j_1 i_1} (\Delta_f \otimes L_{g_{j_\nu}} \otimes \ldots \otimes L_{g_{j_1}} \Delta_f^r h_i)) = 0, \\ &\qquad\qquad\qquad\qquad i = 1, \ldots, q, \quad 0 \leqslant r \leqslant d_i - 1 \end{aligned}$$

because of the commutativity of the tensor product and the definition of d_i. ■

Remark 6

It follows from Lemma 2 that each output index of the feedback system, say $\bar{d}_i$, is greater than or equal to d_i, $i = 1, \ldots, q$; i.e. $\bar{d}_i \geqslant d_i$, $i = 1, \ldots, q$.

Lemma 3

Let $d_i < \infty$, $i = 1, \ldots, p$. The smallest dimension of the linear system into which Σ can be immersed by feedback is greater than or equal to $\max_i \{d_i\}_1{}^q + 1$.

Proof

Suppose that $\dim \bar{\theta} = r$. Because of Lemma 1, $r \geqslant \max \{\bar{k}_i\}_1{}^q$; because of Remark 6, $\max_i \{\bar{k}_i\}_1{}^q \geqslant \max_i \{d_i\}_1{}^q + 1$; hence $r \geqslant \max_i \{\bar{d}_i\} + 1$ or $\max_i \{\bar{d}_i\} \leqslant r - 1$. Moreover, because of Remark 6, $\max_i \{\bar{d}_i\}_1{}^q \geqslant \max_i \{d_i\}_1{}^q$; hence $r \geqslant \max_i \{d_i\}_1{}^q + 1$. ■

Let us define the matrix $\{a_{ij}(x)\}$, $i = 1, \ldots, q$, $j = 1, \ldots, p$, as

$$a_{ij}(x) \triangleq \Delta_f \otimes L_{g_j} \Delta_f{}^{d_i}(h_i) \tag{24}$$

Definition 3

Σ is said to be immersed by feedback with full control into a linear system if there exist α_i and β_{ij}, $i = 1, \ldots, p$, $j = 1, \ldots, p'$, $p' \geqslant p$, such that the resulting feedback system is immersed into a linear one and $\boldsymbol{\beta}(x)$ has full row rank (equal to p).

Theorem 2

Let $d_i < \infty$, $i = 1, \ldots, q$. Σ is immersed by feedback into a linear system if

(i) $\Delta_f \otimes L_{g_{i_\nu}} \otimes \ldots \otimes L_{g_{i_1}} \Delta_f{}^{d_i}(h_i) = 0, \quad i = 1, \ldots, q, \quad \forall \nu \geqslant 2,$
$i_\nu, \ldots, i_1 \in \{1, \ldots, p\}$ (25)

(ii) there exist $\boldsymbol{\alpha}(x)$, $(p \times 1)$, and $\boldsymbol{\beta}(x)$, $(p \times p')$, such that

$$A(x) \cdot \boldsymbol{\beta}(x) = M \quad \text{(a constant } q \times p' \text{ matrix)} \tag{26}$$

$$A(x) \cdot \boldsymbol{\alpha}(x) = \begin{cases} -\Delta_f{}^{d_1+1}(h_1) + \sum\limits_{i=1}^{d_1+1} a^1{}_{d_1+1,\, i} \Delta_f{}^{i-1}(h_1) + \ldots \\ \qquad + \sum\limits_{i=1}^{d_q+1} a^1{}_{d_1+1,\, d_1+\ldots+d_{q-1}-q-1+i} \Delta_f{}^{i-1}(h_q) \\ \vdots \\ -\Delta_f{}^{d_q+1}(h_{\bar{q}}) + \sum\limits_{i=1}^{d_1+1} a^q{}_{n',\, i}\, \Delta_f{}^{i-1}(h_1) + \ldots \\ \qquad + \sum\limits_{i=1}^{d_q+1} a^q{}_{n',\, d_1+\ldots+d_{q-1}+q-1+i} \Delta_f{}^{i-1}(h_q) \end{cases} \tag{27}$$

for a suitable choice of the coefficients $a_{ij}{}^{l}$; where

$$n' = q + \sum_{i=1}^{q} d_i$$

Proof

We shall show that under feedback defined by (26) and (27) the modified system is immersed into a linear one. To this end let us denote by $\tau : \mathbb{R}^N \to \mathbb{R}^{n'} : x \to \tilde{x}$ the following analytic map

$$\begin{aligned}\tau(x) &= (h_1, \dots, \Delta_f{}^{d_1}(h_1), \dots, h_q, \dots, \Delta_f{}^{d_q}(h_q))'\big|_x \\ &= (\tilde{x}_1{}^1, \dots, \tilde{x}^1{}_{d_1+1}, \dots, \tilde{x}_1{}^q, \dots, \tilde{x}^q{}_{d_q+1})'\end{aligned}$$

where the vertical bar $|$ denotes evaluation at x.

It follows that

$$\begin{aligned}
\tilde{x}_1{}^1(t+1) &= h_1\big|_{x(t+1)} = \Delta_f(h_1)\big|_{x(t)} \\
&= \Delta_f(h_1)\big|_{x(t)} \quad \text{(because of the definition of } d_1\text{)} \\
&\vdots \\
\tilde{x}^1{}_{d_1+1}(t+1) &= \Delta_f{}^{d_1}\, h_1\big|_{x(t+1)} = \Delta_f \Delta^{d_1}(h_1)\big|_{x(t)} \\
&= \Delta_f{}^{d_1+1}(h_1)\big|_{x(t)} + \sum_{i=1}^{p} u_i \Delta_f \otimes L_{g_i} \Delta_f{}^{d_1}(h_1)\big|_{x(t)} \quad \text{(because of (i))} \\
&= \Delta_f{}^{d_1+1}(h_1)\big|_{x(t)} + \sum_{i=1}^{p} \Big(\alpha_i(x) + \sum_{j=1}^{p'} \beta_{ij} v_j\Big) \Delta_f \otimes L_{g_i} \Delta_f{}^{d_1}(h_1)\big|_{x(t)} \\
&= \sum_{i=1}^{d_1+1} a^1{}_{d_1+1,i} \Delta_f{}^{i-1}(h_1)\big|_{x(t)} + \cdots \\
&\quad + \sum_{i=1}^{l_q+1} a^1{}_{d_1+1,\, d_1+\dots+d_{q-1}+q-1-i} \Delta_f{}^{i-1}(h_q)\big|_{x(t)} \\
&\qquad + \sum_{j=1}^{p} m_{1j} v_j \quad \text{(because of (ii))} \\
&\vdots \\
\tilde{x}_1{}^q(t+1) &= h_q\big|_{x(t+1)} = \Delta_{f+\Sigma g_i u_i}(h_q) = \Delta_f(h_q)\big|_{x(t)} \\
&\vdots \\
\tilde{x}^q{}_{d_q+1}(t+1) &= \Delta_f{}^{d_q+1}(h_q)\big|_{x(t+1)} \\
&= \Delta_f{}^{d_q+1}(h_q)\big|_{x(t)} + \sum_{i=1}^{p} u_i \Delta_f \otimes L_{g_i} \Delta_f{}^{d_q}(h_q)\big|_{x(t)} \quad \text{(because of (i))} \\
&= \sum_{i=1}^{d_1+1} a^q{}_{n',i}\; \Delta_f{}^{i-1}(h_1)\big|_{x(t)} + \cdots \\
&\quad + \sum_{i=1}^{l_q+1} a^q{}_{n',\, d_1+\dots+d_q+q-1+i} \Delta_f{}^{i-1}(h_q)\big|_{x(t)} \\
&\qquad + \sum_{j=1}^{p} m_{qj} v_j \quad \text{(because of (ii))}
\end{aligned}$$

and the linear system into which Σ is immersed assumes the form

$$A=\begin{bmatrix}
0 & 1 \quad 0 \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \\
\vdots & & \vdots & \vdots & & & & & & & \\
0 & \cdots\cdots\; 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \\
a^1_{d_1+1,1} & \cdots & a^1_{d_1+1,d_1+1} & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & a^1_{d_1+1,n'} & \\
0 & \cdots & 0 & 0 & 1 \;\cdots & 0 & \cdots & 0 & \cdots & 0 & \\
\vdots & & \vdots & \vdots & \ddots & & & & & & \\
0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \\
a^2_{d_1+d_2+2,1} & \cdots & a^2_{d_1+d_2+2,d_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a^2_{d_1+d_2+2,n'} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \;\cdots & 0 & \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots & \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \\
a^q_{n',1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a^q_{n',n'}
\end{bmatrix} \tag{28'}$$

$$B=\begin{bmatrix}
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
m_{11} & \cdots & m_{1p'} \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
m_{21} & \cdots & m_{2p'} \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
m_{q1} & \cdots & m_{qp'}
\end{bmatrix} \tag{28''}$$

$$C=\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
0 & & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots\; 0 & 1 & 0 \;\cdots & 0
\end{bmatrix} \tag{28'''}$$

■

We note that if the functions $\{h_1, \ldots, \Delta_f^{d_1}(h_1), \ldots, h_q, \ldots, \Delta_f^{d_q}(h_q)\}$ generate a finite-dimensional vector space over $\mathbb{R}$, then sufficient conditions for the immersion under feedback are expressed by (25) and (26) with (27) satisfied with $\boldsymbol{\alpha}(x) \equiv \mathbf{0}$. Moreover, this hypothesis of finite dimensionality is equivalent to assuming that at least for an index $k \in \{1, \ldots, q\}$ there exists $l_k \leqslant d_k$ such that

$$\Delta_f^{l_k+n}(h_k) = \sum_{i=0}^{d_1} c_i^{1,n}\, \Delta_f^i(h_1) + \ldots + \sum_{i=0}^{l_k} c_i^{k,n}\, \Delta_f^i(h_k) + \ldots + \sum_{i=0}^{d_q} c_i^{q,n}\, \Delta_f^i(h_q), \quad \forall n \geqslant 1$$

Following Claude *et al.* (1983), let us define with respect to the linear system Σ_L given by (24) and (25)

$$\mathscr{V}_i \triangleq \bigcap_{i=1}^{d_i} \mathbf{c}_i A^j, \quad i = 1, \ldots, q$$

$$\mathscr{I} \triangleq \bigcap_{i=1}^{q} \bigcap_{j=0}^{n-1} \mathbf{c}_i A^j$$

where $\mathbf{c}_i$ denotes the ith row of the matrix C.

Note that the linear system Σ_L with matrices given by (28) is such that the unobservables $\mathscr{I}$ coincide with the intersection of the subspaces $\mathscr{V}_i$, $i = 1, \ldots, q$, in particular in this case $\mathscr{I} = \{0\}$. Theorem 3 shows that conditions (i) and (ii) of Theorem 2 are necessary for the immersion with full control into a linear system which enjoys the property mentioned.

Theorem 3

Let $d_i < \infty$, $i = 1, \ldots, q$; then the conditions (i) and (ii) of Theorem 2 hold if Σ is immersed by feedback with full control into a linear system such that

$$\text{(i)} \qquad \bigcap_{i=1}^{q} \mathscr{V}_i = \mathscr{I}$$

Proof

By hypothesis there exist $\boldsymbol{\alpha}(x)$ and $\boldsymbol{\beta}(x)$ with full row rank such that

$$\text{(i)} \qquad \dim \bar{\theta} = k \geqslant \max_i \{\bar{d}_i\} + 1$$

$$\text{(ii)} \quad \Delta_{f-\Sigma\alpha_i g_i} \otimes L_{(g\beta)_k} \Delta^{r_l}{}_{f+\Sigma\alpha_i g_i}(h_l) = c_{kl}{}^{r_l}, \quad k = 1, \ldots, p'; \quad l = 1, \ldots, q, \quad \forall r_l$$

$$\text{(iii)} \quad \Delta_{f+\Sigma\alpha_i g_i} \otimes L_{(g\beta)_{i_\nu}} \otimes \ldots \otimes L_{(g\beta)_{i_1}} \Delta^{r_l}{}_{f+\Sigma\alpha_i g_i}(h_l) = 0, \quad l = 1, \ldots, q, \quad \forall \nu \geqslant 2; \quad \forall r_l, \quad i_1, \ldots, i_\nu \in \{1, \ldots, p'\}$$

When $r_l = d_l$ we have

$$\begin{aligned}
\text{(ii)} \quad & \Delta_{f+\Sigma\alpha_i g_i} \otimes L_{(g\beta)_k} \Delta^{d_l}{}_{f+\Sigma\alpha_i g_i}(h_l) = c_{kl} \\
& \overset{\text{(Lemma 2)}}{=} \Delta_{f+\Sigma\alpha_i g_i} \otimes L_{(g\beta)_k} \Delta_f^{d_l}(h_l) \\
& = \sum_{j=1}^{p} \beta_{jk} \Delta_{f+\Sigma\alpha_i g_i} \otimes L_{g_j} \Delta_f^{d_l}(h_l) \\
& = \sum_{j=1}^{p} \beta_{jk} \Big(\Delta_f \otimes L_{g_j} \Delta_f^{d_l}(h_l) + \sum_{\sigma=1}^{p} \sum_{k_1 > k_2 > \ldots > k_\sigma = 1}^{p} \sum_{n_{k_1}, \ldots, n_{k_\sigma} \geqslant 1} \\
& \quad \times \frac{\alpha_{k_1}{}^{n_{k_1}} \ldots \alpha_{k_\sigma}{}^{n_{k_\sigma}}}{n_{k_1}! \ldots n_{k_\sigma}!} \Delta_f \otimes L_{g_{k_1}}{}^{\otimes n_{k_1}} \otimes \ldots \otimes L_{g_{k_\sigma}}{}^{\otimes n_{k_\sigma}} \otimes L_{g_j} \Delta_f^{d_l}(h_l) \Big)
\end{aligned} \qquad (29)$$

(iii) $\Delta_{f+\Sigma\alpha_i g_i}\otimes L_{(g\beta)_{i_\nu}}\otimes\dots\otimes L_{(g\beta)_{i_1}}\Delta^{d_l}{}_{f+\Sigma\alpha_i g_i}(h_l)=0$

$$\overset{\text{(Lemma 2)}}{=}\ \Delta_{f+\Sigma\alpha_i g_i}\otimes\left(\sum_{j_\nu=1}^{p}\beta_{j_\nu i_\nu}L_{g_{j_\nu}}\right)\otimes\dots\otimes\left(\sum_{j_1=1}^{p}\beta_{j_1 i_1}L_{g_{j_1}}\right)\Delta_f{}^{d_l}(h_l)$$

$$=\sum_{j_1,\dots,j_\nu=1}^{p}\beta_{j_1 i_1}\dots\beta_{j_\nu i_\nu}\Delta_{f+\Sigma\alpha_i g_i}\otimes L_{g_{j_\nu}}\otimes\dots\otimes L_{g_{j_1}}\Delta_f{}^{d_l}(h_l)$$

$$=\sum_{j_1,\dots,j_\nu=1}^{p}\beta_{j_1 i_1}\dots\beta_{j_\nu i_\nu}(\Delta_f\otimes L_{g_{j_\nu}}\otimes\dots\otimes L_{g_{j_1}}\Delta_f{}^{d_l}(h_l)$$

$$+\sum_{\sigma=1}^{p}\ \sum_{k_1>k_2>\dots>k_\sigma=1}^{p}\ \sum_{n_{k_1},\dots,n_{k_\sigma}\geqslant 1}\frac{\alpha_{k_1}{}^{n_{k_1}}\dots\alpha_{k_\sigma}{}^{n_{k_\sigma}}}{n_{k_1}!\dots n_{k_\sigma}!}$$

$$\times\Delta_f\otimes L_{g_{k_1}}{}^{\otimes n_{k_1}}\otimes\dots\otimes L_{g_{k_\sigma}}{}^{\otimes n_{k_\sigma}}\otimes L_{g_{j_\nu}}\otimes\dots\otimes L_{g_{j_1}}\Delta_f{}^{d_l}(h_l))=0 \quad (30)$$

Notice now that for any choice of $\{j_\nu,\dots,j_1\}$, $j_l\in\{1,\dots,p\}$, $l=1,\dots,\nu$, there exist $s_1,\dots,s_p\in N$ such that

$$L_{g_{j_\nu}}\otimes\dots\otimes L_{g_{j_1}}=L_{g_1}{}^{\otimes s_1}\otimes\dots\otimes L_{g_p}{}^{\otimes s_p}$$

Define now

$$E(s_1,\dots,s_p)\triangleq\Delta_f\otimes L_{g_1}{}^{\otimes s_1}\otimes\dots\otimes L_{g_p}{}^{\otimes s_p}\ \Delta_f{}^{d_l}(h_l)$$

so that

$$E(s_1,\dots,s_p)=\sum_{\substack{p_1\geqslant s_1\\ \vdots\\ p_p\geqslant s_p}}\frac{(-\alpha_1)^{p_1-s_1}\dots(-\alpha_p)^{p_p-s_p}}{(p_1-s_1)!\dots(p_p-s_p)!}\times\Bigg[E(p_1,\dots,p_p)$$

$$+\sum_{\sigma\geqslant 1}\ \sum_{k_1>\dots>k_\sigma=1}^{p}\ \sum_{n_{k_1},\dots,n_{k_\sigma}\geqslant 1}E(n_{k_1}+p_1,\dots,n_{k_p}+p_p)\Bigg]$$

Equation (30) being verified for any $\nu\geqslant 2$, $i_1,\dots,i_\nu\in\{1,\dots,p'\}$, it follows, from the full row rank of $\boldsymbol{\beta}$, that any term in (30) (for any choice of $j_1,\dots,j_\nu\in\{1,\dots,p\}$) is equal to zero; hence $E(s_1,\dots,s_p)=0$ for all $s_i\geqslant 0$, $i=1,\dots,p$. Equation (30) being verified for $l=1,\dots,p$, the conditions (25) hold true. By considering the equality (29), because of (25), it follows that condition (26) is satisfied.

As far as (27) is concerned, first of all note that because of (26) and the full row rank of $\boldsymbol{\beta}$ it follows from (29) that $\bar{d}_i=d_i$, $i=1,\dots,q$. Moreover, condition (i), for the linear system into which Σ_F is immersed, implies

$$c_l A^{d_l+1}=\sum_{i=0}^{d_1}a_i{}^1\,c_1A^i+\dots+\sum_{i=0}^{d_q}a_i{}^q c_q A^i$$

Hence

$$\Delta_f{}^{d_l+1}(h_l)=\Delta_{\bar f}{}^{\bar d_l+1}(h_l)=\sum_{i=0}^{d_1}a_i{}^1\,\Delta_f{}^i(h_1)+\dots+\sum_{i=0}^{d_q}a_i{}^q\,\Delta_f{}^i(h_q),\quad l=1,\dots,q$$

Finally it follows from Lemma 2 and conditions (25) that

$$\begin{aligned}\Delta_{\tilde f}{}^{d_l+1}(h_l) &= \Delta_{\tilde f}\Delta_{\tilde f}{}^{d_l}\, h_l\\ &= \Delta_f{}^{d_l+1}\, h_l + \sum_{i=1}^{p} \alpha_i \Delta_f \otimes L_{g_i} \Delta_f{}^{d_l}\, h_l\,; \quad l=1, \ldots, q\end{aligned}$$

Recalling from Lemma 2 that $\Delta_{\tilde f}{}^{i}(h_j) = \Delta_f{}^{i}(h_j)$, $j=1, \ldots, p$, for all $i \leqslant d_j$, the last two equalities imply (27). ■

Remark 7

We note that an algebraic condition equivalent to (i) of Theorem 3 is that there exists a basis for $\bar{\theta}$ chosen as in Lemma 1 such that $l_i \leqslant d_i + 1$; $i = 1, \ldots, q$. It is easy to verify that this assumption, instead of (i), implies (27).

As far as immersion by feedback into a linear controllable system is concerned we have the following corollary.

Corollary 1

Assume that $d_i < \infty$, $i=1, \ldots, q$. Properties (i) and (ii) of Theorem 2 and the full row rank of M (in (26)) imply the immersion by feedback into a linear controllable system. The immersion by feedback with full control into a linear controllable system implies (i) and (ii) of Theorem 2.

A particularly interesting situation happens when Σ has the same number of inputs and outputs, $p=q$, and $A(x)$ is invertible. Under those hypotheses condition (i) of Theorem 2 is necessary and sufficient for the immersion by feedback into a linear controllable system. More precisely we have the following.

Corollary 2

Assume that $d_i < \infty$, $i=1, \ldots, q$ and that $A(x)$ be invertible. Σ is immersed by feedback with full control into a linear controllable system which enjoys the property (i) of Theorem 3 if and only if the conditions (i) of Theorem 2 are verified.

Proof

The necessity follows from Theorem 3. As far as the sufficiency is concerned, let

$$\tau(x) = \tilde{x} = \left.\begin{bmatrix} h_1 \\ \vdots \\ \Delta_f{}^{d_1}(h_1) \\ \vdots \\ h_q \\ \vdots \\ \Delta_f{}^{d_q}(h_q) \end{bmatrix}\right|_x = \begin{bmatrix} \tilde{x}_1{}^1 \\ \vdots \\ \tilde{x}^1{}_{d_1+1} \\ \vdots \\ \tilde{x}_1{}^q \\ \vdots \\ \tilde{x}^q{}_{d_q+1} \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ z_{\Sigma d_i + q} \end{bmatrix}$$

the feedback defined by

$$\boldsymbol{\beta}(x) = [A(x)]^{-1} \tag{31}$$

$$\boldsymbol{\alpha}(x) = [A(x)]^{-1} \begin{bmatrix} -\tilde{x}_{d_1+1} + \sum\limits_{i=1}^{k} a_i{}^1 z_i \\ \vdots \\ -\tilde{x}_{d_q+1} + \sum\limits_{i=1}^{k} a_i{}^q z_i \end{bmatrix} \tag{32}$$

where $k=\sum_{i=1}^{q} d_i+q$, satisfy (26) and (27) for any choice of a_i^l, $i=1, \ldots, k$, $l=1, \ldots, q$. The resulting feedback system is hence immersed into a linear system defined by (28) which is controllable for the assumption $M=I$ in (26). ■

It is interesting to note that because of the arbitrariness of the coefficients a_i^l in (32), this choice can be performed in order to satisfy any design requirement by means of assignment of the dynamics.

In conclusion we note that by choosing

$$\alpha(x)=[A(x)]^{-1}\begin{bmatrix} -\tilde{x}_{d_1+1}+\sum_{i=1}^{d_1+1} a_i^1 \tilde{x}_i^1 \\ \vdots \\ -\tilde{x}_{{}_qd+1}+\sum_{i=1}^{d_q+1} a_i^q \tilde{x}_i^q \end{bmatrix} \tag{33}$$

the system into which Σ is immersed by feedback defined by (28) results to be input–output decoupled (i.e. each input channel u_i affects only the corresponding y_i output). Moreover the dynamics of each input–output pair can be arbitrarily assigned. Hence the feedback defined by (31)–(33) makes the non-linear system modified by feedback decoupled, with any linear dynamics possible for any input–output pair.

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