Systems & Control Letters 2 (1983) 313–320
North-Holland Publishing Company

February 1983

Functional output ε-controllability for linear systems on Hilbert spaces

A. GERMANI

Istituto di Analisi dei Sistemi ed Informatica del C.N.R., Via Buonarroti 12, 00185 Roma, Italy

S. MONACO

Istituto di Automatica, Università di Roma, Via Eudossiana 18, 00184 Roma, Italy

Received 27 November 1982

This paper deals with the problem of the functional output ε-controllability of a linear system whose state space is a real separable Hilbert space. In particular a condition which assures such a property is found.

Keywords: Functional controllability, Infinite dimensional systems.

Introduction

The functional output controllability, which concernes the control of the output of a dynamical system over an interval of time, was firstly studied by Brockett and Mesarovic in [1] where the authors developped an algebraic test condition for the class of linear, finite dimensional, dynamical systems.

The functional output controllability of linear finite dimensional dynamical systems, which turns out to be the dual property of invertibility, was subsequently investigated by Sain and Massey in [2] and Rosenbrock in [3].

A similar concept, that of perfect partial output controllability defined as the capability of reproducing by the output of a dynamical system a class of trajectories belonging to given output subspaces, was studied by Basile and Marro in [4], [5]. The approach developped therein is based on the concept of controlled and conditioned invariant subspaces. Following a similar approach, more recently Basile and Hamano in [6] introduced and studied the concept of order of controllability for the output of a linear dynamical system; the order of controllability turns out to characterize the rapidity of the possible output movements.

The aim of this paper is to study the notion of functional output ε-controllability introduced by Hiris and Megan in [7], and by the authors themselves in [8], for the class of linear differential systems on Hilbert spaces. The notion of functional output ε-controllability turns out to be a slight generalization of the functional output controllability in the same way as the ε-controllability is a generalization of the notion of controllability, [9]. The functional output ε-controllability is related to the capability of the input to drive the output as close as possible to any fixed output trajectory belonging to a given functional space. This notion plays a central role in many control problems in particular when a fixed class of output-trajectories is fixed and constraints on the admissible input functions are given.

In this paper a first sufficient condition is given for the functional output ε-trajectory controllability when no other constraints on the input and output functions are assumed but the finiteness of the associate energy. We like to stress that the main difficulty to state the result here presented is due to the fact that no special assumptions on the dynamical operator are assumed.

Some interesting results on the subject are given in [7] where the dynamical operator of the system is assumed to be bounded.

The paper is divided into two sections; the following one contains the preliminaries, the second one is devoted to present the announced result.

1. Preliminaries

We will consider linear control systems described by

0167-6911/83/0000–0000/$03.00

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

$$y(t) = Cx(t), \quad x(0) = \hat{x}, \tag{1.2}$$

where

$$x(t) \in \mathcal{H}_x; \quad u(t) \in \mathcal{H}_u; \quad y(t) \in \mathcal{H}_y,$$

with $\mathcal{H}_x$, $\mathcal{H}_u$ and $\mathcal{H}_y$ separable real Hilbert spaces endowed with inner products and norms denoted by

$$[\cdot, \cdot]_x, \quad [\cdot, \cdot]_u, \quad [\cdot, \cdot]_y$$

and

$$|\cdot|_x, \quad |\cdot|_u, \quad |\cdot|_y$$

respectively. $A : \mathcal{H}_x \to \mathcal{H}_x$ is the infinitesimal generator of a strongly continuous semigroup of operators $\{\Gamma(t), t \geqslant 0\}$; $B : \mathcal{H}_u \to \mathcal{H}_x$ and $C : \mathcal{H}_x \to \mathcal{H}_y$ are bounded linear operators.

Moreover we will denote by W_α^T the Hilbert space $L_2([0, T]; \mathcal{H}_\alpha)$ endowed with the usual inner product

$$[\varphi, \psi]_{W_\alpha^T} \triangleq \int_0^T [\varphi(t), \psi(t)]_\alpha \, \mathrm{d}t \tag{1.3}$$

and the associate norm denoted by $\|\cdot\|_{W_\alpha^T}$.

It is well known, [10], that under the above assumptions the equation (1.1) has to be interpreted as a representation of the following integral equation:

$$[x(t), a] = [\hat{x}, a] + \int_0^t [x(s), A^* a] \, \mathrm{d}s + \int_0^t [Bu(s), a] \, \mathrm{d}s \tag{1.4}$$

for $t \in [0, T]$ and any $a \in \mathcal{D}(A^*)$ (the domain of the adjoint A^* of A). Moreover any solution of (1.1) in W_x^T is a solution of the equation

$$x = x_0 + Lu \tag{1.5}$$

where

$$x_0(t) = \Gamma(t)\hat{x}, \tag{1.6}$$

$L : W_u^T \to W_x^T$ is defined by $y = Lu$:

$$y(t) = \int_0^t \Gamma(t - \tau) Bu(\tau) \, \mathrm{d}\tau, \tag{1.7}$$

and the integral in (1.7) has to be interpreted as a Pettis integral, [10].

Let us now give the following definition, [7,8]:

Definition 1. The linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$, $\mathcal{H}_1$ and $\mathcal{H}_2$ separable Hilbert spaces, is said to be *ε-controllable in* $y \in \mathcal{H}_2$ if for any $\varepsilon > 0$ there exists $z_{\varepsilon,y} \in \mathcal{H}_1$ such that

$$\|Tz_{\varepsilon,y} - y\|_{\mathcal{H}_2} < \varepsilon. \tag{1.8}$$

It will be said to be *ε-controllable* if it is ε-controllable in any $y \in \mathcal{H}_2$. Moreover we define T to be *controllable* if (1.8) is true with $\varepsilon = 0$.

In the sequel when $T = CL : W_u^T \to W_y^T$, L defined by (1.7), we will refer to the controllability property of Definition 1 as functional output ε-controllability of the system (1.1), (1.2).

2. Main result

In this section we state the announced result about the functional output ε-controllability of the system (1.1), (1.2). At first let us consider the case of finite dimensionality of $\mathcal{H}_y$, say q, so that C can be represented as

$$y = Cx = \sum_{i=1}^{q} \alpha_i [x, \varphi_i] \psi_i \tag{2.1}$$

where $\{\varphi_i\}_1^\infty$ and $\{\psi_i\}_1^q$ are orthonormal bases for $\mathcal{H}_x$ and $\mathcal{H}_y$ respectively.

With the introduced notation, and denoting $N(C) \subset \mathcal{H}_x$ the null space of the linear operator C, one has

$$N(C) = \mathrm{span}\{\varphi_i, i = q+1, q+2, \ldots\}. \tag{2.2}$$

Let us denote by $\pi : \mathcal{H}_x \to \mathcal{H}_x$ the projection operator on $N(C)^\perp$: the orthogonal complement of $N(C)$; one has:

$$\pi x = \sum_{i=1}^{q} [x, \varphi_i] \varphi_i. \tag{2.3}$$

Finally let us denote by $\mathcal{R}_B$ and $\mathcal{I}_C$ the reachable set and the set of unobservables of the system (1.1), (1.2) respectively; one has

$$\mathcal{R}_B \triangleq \overline{\bigcup_{t \geqslant 0} \Gamma(t) B \mathcal{H}_u} \tag{2.4a}$$

and

$$\mathcal{I}_C \triangleq \overline{\bigcup_{t \geqslant 0} \Gamma^*(t) C^* \mathcal{H}_u}^{\perp} \tag{2.4b}$$

where the bar stands for closure and $\perp$ for the orthogonal complement.

Under the above positions:

Lemma 1. *Let $(CB)^*$ be the adjoint operator of CB. If $CB(CB)^* : \mathcal{H}_y \to \mathcal{H}_y$, is full rank, then the linear operator $CL_\Delta : W_u^\Delta \to \mathcal{H}_y$ given by*

$$y = CL_\Delta u = \int_0^\Delta C\Gamma(\Delta - \tau) Bu(\tau)\, d\tau \tag{2.5}$$

(i) *has the decomposition*

$$CL_\Delta = Z_\Delta W_\Delta \tag{2.6}$$

with

$W_\Delta : W_u^\Delta \to \mathcal{H}_y$;

$W_\Delta W_\Delta^* = I$ *in* $\mathcal{H}_y$;

$Z_\Delta : \mathcal{H}_y \to \mathcal{H}_y$ *positive definite and selfadjoint*;

(ii) *is controllable for any* $\Delta > 0$.

To prove (i) let us consider the polar decomposition of $(CL_\Delta)^*$, [11], given by

$$(CL_\Delta)^* = U_\Delta \cdot Z_\Delta \tag{2.7}$$

where $U_\Delta : \mathcal{H}_y \to W_u^\Delta$ is a partial isometry, i.e. $U_\Delta^* U_\Delta$ is a projection on $\mathcal{H}_y$; and $Z_\Delta : \mathcal{H}_y \to \mathcal{H}_y$ is nonnegative definite and selfadjoint.

It follows that the operator $CL_\Delta(CL_\Delta)^* : \mathcal{H}_y \to \mathcal{H}_y$,

$$CL_\Delta(CL_\Delta)^* = \int_0^\Delta C\Gamma(\Delta - \tau) BB^*\Gamma^*(\Delta - \tau) C^*\, d\tau,$$

has full rank. For, suppose the existence of $y \in \mathcal{H}_y$ such that

$$\left[y, \int_0^\Delta C\Gamma(t) BB^*\Gamma^*(t) C^*\, dt y \right]_{\mathcal{H}_y} = 0$$

$$\Leftrightarrow \int_0^\Delta \| C\Gamma(t) By \|^2\, dt = 0.$$

Then $CBy = 0$ which contradicts the hypothesis.

Since

$$CL_\Delta(CL_\Delta)^* = Z_\Delta U_\Delta^* U_\Delta Z_\Delta$$

is a full rank operator and $U_\Delta^* U_\Delta$ a projection on $\mathcal{H}_y$, it follows that

$$U_\Delta^* U_\Delta = I \tag{2.8}$$

and

$$\Lambda_{i_{\min}}^\Delta \triangleq \min_{i=1,\dots,q} \left\{ \text{eigenvalues of} \int_0^\Delta C\Gamma(t) BB^*\Gamma^*(t) C^*\, dt \right\} > 0. \tag{2.9}$$

(2.7), (2.8) and (2.9) imply the existence of the decomposition (2.6) with $W_\Delta = U_\Delta^*$.

To prove (ii) it is sufficient to note, from (2.9), that

$$\left\| \left(\int_0^\Delta C\Gamma(t) BB^*\Gamma^*(t) C^*\, dt \right)^{-1} \right\|_{\mathcal{H}_y} = \frac{1}{\lambda_{i_{\min}}^\Delta} < \infty.$$

Hence the input function

$$u(\tau) = B^*\Gamma^*(\Delta - \tau) C^* \cdot \left(\int_0^\Delta C\Gamma(\xi) BB^*\Gamma^*(\xi) C^* d\xi \right)^{-1} y,$$

which satisfies (2.5) for any $y \in \mathcal{H}_y$, belongs to W_u^Δ. □

Theorem 1. *The system* (1.1), (1.2) *has the functional output ε-controllability property if there exists a bounded linear operator* $P : \mathcal{H}_u \to \mathcal{H}_u$ *such that*:

$$\text{(i) } CBPP^*B^*C^* \text{ is full rank}, \tag{2.10}$$

$$\text{(ii) } (I - \pi)\mathcal{R}_{BP} \subset \mathcal{I}_C \tag{2.11}$$

with $\mathcal{R}_{BP}$ *and* $\mathcal{I}_C$ *defined by* (2.4).

Proof. First of all note that the linear operator $CL_{\Delta,P} : W_u^\Delta \to \mathcal{H}_y$ given by

$$y = CL_{\Delta,P} u = \int_0^\Delta C\Gamma(\Delta - \tau) BPu(\tau)\, d\tau \tag{2.12}$$

satisfies the hypothesis of Lemma 1; hence it is controllable for any $\Delta > 0$. Moreover $CL_{\Delta,P}$ admits the decomposition (2.6):

$$CL_{\Delta,P} = T_\Delta V_\Delta \tag{2.13}$$

with $V_\Delta : W_u^\Delta \to \mathcal{H}_y$; $V_\Delta V_\Delta^* = I$ in $\mathcal{H}_y$; $T_\Delta : \mathcal{H}_y \to \mathcal{H}_y$ positive definite and selfadjoint.

Let $\{\phi_i^\Delta;\ i = 1,\dots,q\}$ be an orthonormal basis in $\mathcal{H}_y$ such that $T_\Delta \phi_i^\Delta = \lambda_i^\Delta \phi_i^\Delta$; and let us introduce the linear bounded operator $J_\Delta : \mathcal{H}_y \to W_u^\Delta$ defined by

$$J_\Delta y = \sum_{i=1}^q \frac{[y, \phi_i^\Delta]_y}{\lambda_i^\Delta} V_\Delta^* \phi_i^\Delta \tag{2.14}$$

where V_Δ^* denotes the adjoint operator of V_Δ. The boundedness of J_Δ follows from

$$\|J_\Delta y\|_{W_u^\Delta} \leqslant \frac{1}{\lambda_{i_{\min}}^\Delta}\|y\|_y$$

with $\lambda_{i_{\min}}^\Delta$ defined as in (2.9) with respect to $CL_{\Delta,P}$.

Note that

$$CL_{\Delta,P}J_\Delta y = T_\Delta V_\Delta J_\Delta y$$
$$= T_\Delta V_\Delta \sum_{i=1}^{q} \frac{[y, \phi_i^\Delta]_y}{\lambda_i^\Delta} V_\Delta^* \phi_i^\Delta = y.$$

Hence

$$CL_{\Delta,P}J_\Delta = I, \tag{2.15}$$

so that $J_\Delta y$ is the input which forces the output from zero to y at the time Δ.

Let us consider the decomposition of the time interval $[0, T]$,

$$\{t_0 = 0, t_1, \dots, t_n = T\},$$

such that

$$\Delta \triangleq \frac{T}{n} = t_i - t_{i-1}, \quad i = 1, \dots, n.$$

The proof will be achieved as soon as we prove that for any given $\varepsilon > 0$ there exists a partition of the time interval $[0, T]$ into n_ε subintervals of Δ_ε amplitude and a sequence of input functions, defined on each subinterval, such that the corresponding output function differs for less than ε, in the W_y^T norm, from the fixed one.

At this end let $y \in W_y^T$ be a uniform continuous fixed output trajectory on $[0, T]$; this choice does not imply loss of generality because of the denseness of the uniformly continuous functions in W_y^T. Now define the sequence

$$x^*(0) = 0,$$
$$x^*(t_{i-1} + \theta) = \Gamma(\theta)x^*(t_{i-1}) + L_\theta J_\Delta(y(t_i) - C\Gamma(\Delta)x^*(t_{i-1})), \tag{2.16}$$

which has the property

$$Cx^*(t_i) = Cx^*(t_{i-1} + \Delta) = y(t_i).$$

Let us define the input sequence $\{u_n\}$ where

$$u_n(t) \triangleq J_\Delta(y(t_i) - C\Gamma(\Delta)x^*(t_{i-1}))(t),$$
$$t \in [t_{i-1}, t_i).$$

We will show that for any given $\varepsilon > 0$, there exists N_ε such that for any $N > N_\varepsilon$ one has

$$\|y - y_N\|_{W_y^T} < \varepsilon$$

where y is the fixed output function and y_N is the output corresponding to the input u_N.

We have

$$\|y - y_N\|_{W_y^T}^2 = \int_0^T \|y(t) - y_N(t)\|_y^2 \, dt$$
$$= \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|y(t) - y_N(t)\|_y^2 \, dt$$
$$= \sum_{i=1}^{N} \int_0^\Delta \|y(t_{i-1} + \theta) - C\Gamma(\theta)x^*(t_{i-1}) - CL_{\theta,P}J_\Delta(y(t_i) - C\Gamma(\Delta)x^*(t_{i-1}))\|_y^2 \, d\theta$$
$$\leqslant 4\sum_{i=1}^{N} \int_0^\Delta \|y(t_{i-1} + \theta) - y(t_{i-1})\|_y^2 \, d\theta$$
$$+ 4\sum_{i=1}^{N} \int_0^\Delta \|y(t_{i-1}) - C\Gamma(\theta)x^*(t_{i-1})\|_y^2 \, d\theta$$
$$+ 4\sum_{i=1}^{N} \int_0^\Delta \|CL_{\theta,P}J_\Delta(y(t_i) - y(t_{i-1}))\|_y^2 \, d\theta$$
$$+ 4\sum_{i=1}^{N} \int_0^\Delta \|CL_{\theta,P}J_\Delta(y(t_{i-1}) - C\Gamma(\Delta)x^*(t_{i-1}))\|_y^2 \, d\theta. \tag{2.17}$$

We will now examine one by one the four terms in the last inequality of (2.17).

For the first term we have

$$4\sum_{i=1}^{N} \int_0^\Delta \|y(t_{i-1} + \theta) - y(t_{i-1})\|_y^2 \, d\theta \leqslant \frac{\varepsilon}{4} \tag{2.18}$$

as soon as we choose $\Delta \leqslant \Delta'$ small enough in order that

$$\|y(t) - y(\tau)\|_y < \frac{\sqrt{\varepsilon}}{4\sqrt{T}}, \tag{2.19}$$
$$|t - \tau| \leqslant \Delta'.$$

The existence of such a Δ' is assured by the uniform continuity of y in $[0, T]$. Hence (2.18) is true for any $N' \geqslant T/\Delta'$.

As far as the second term is concerned we have

$$4\sum_{i=1}^{N} \int_0^\Delta \|C(I - \Gamma(\theta))x^*(t_{i-1})\|_y^2 \, d\theta$$

$$= 4\sum_{i=1}^{N}\int_0^{\Delta}\|C(I-\Gamma(\theta))\pi x^*(t_{i-1})$$
$$+ C(I-\Gamma(\theta))(I-\pi)x^*(t_{i-1})\|_y^2\,\mathrm{d}\theta$$
$$= 4\sum_{i=1}^{N}\int_0^{\Delta}\|C(I-\Gamma(\theta))\pi x^*(t_{i-1})\|_y^2\,\mathrm{d}\theta \quad (2.20)$$

where the last equality holds because of (2.11).

Moreover,

$$4\sum_{i=1}^{N}\int_0^{\Delta}\Bigg[\sum_{k=1}^{q}\alpha_k\big[(I-\Gamma(\theta))\pi x^*(t_{i-1}),\varphi_k\big]_x\psi_k,$$
$$\sum_{l=1}^{q}\alpha_l\big[(I-\Gamma(\theta))\pi x^*(t_{i-1}),\varphi_l\big]_x\psi_l\Bigg]_y\,\mathrm{d}\theta$$
$$= 4\sum_{i=1}^{N}\sum_{k=1}^{q}\alpha_k\int_0^{\Delta}\big[(I-\Gamma(\theta))\pi x^*(t_{i-1}),\varphi_k\big]_x^2\,\mathrm{d}\theta \quad (2.21)$$

where we have used for the operator C its representation (2.1).

To go ahead we need to state the following properties:

$$\text{(A)}\quad \|L_{\Delta,P}J_\Delta\|_{\mathcal{L}(\mathcal{K}_y,\mathcal{K}_x)} \leqslant M < \infty, \quad (2.22)$$

$$\text{(B)}\quad L_{\Delta,P}J_\Delta C = \pi. \quad (2.23)$$

(A) follows from

$$\|L_{\Delta,P}J_\Delta\|_{\mathcal{L}(\mathcal{K}_y,\mathcal{K}_x)}$$
$$= \sup_{\|y\|=1}\|L_{\Delta,P}J_\Delta y\|_x$$
$$= \sup_{\|y\|=1}\left\|\int_0^{\Delta}\Gamma(\Delta-\tau)BP\sum_{k=1}^{q}\frac{[y,\phi_k^\Delta]_y}{\lambda_k^\Delta}\cdot(V^*\phi_k^\Delta)(\tau)\,\mathrm{d}\tau\right\|_x$$
$$\leqslant \sup_{\|y\|=1}\int^{\Delta}\|\Gamma(\Delta-\tau)\|_{\mathcal{L}(\mathcal{K}_x,\mathcal{K}_x)}\cdot\|BP\|_{\mathcal{L}(\mathcal{K}_u,\mathcal{K}_x)}\cdot\sum_{k=1}^{q}\frac{|[y,\phi_k^\Delta]_y|}{\lambda_k^\Delta}\cdot\|(V^*\phi_k^\Delta)(\tau)\|_u\,\mathrm{d}\tau$$
$$\leqslant M_1\sum_{k=1}^{q}\frac{1}{\lambda_k^\Delta}\int_0^{\Delta}\|(V^*\phi_k^\Delta)(\tau)\|_u\,\mathrm{d}\tau$$
$$\leqslant M_1 q\frac{\sqrt{\Delta}}{\lambda_{i_{\min}}^\Delta} \quad (2.24)$$

with M_1 a suitable constant. For the last inequality we have used the Schwartz inequality.

Taking into account that

$$\lim_{\Delta\downarrow 0}\frac{\left(\lambda_{i_{\min}}^\Delta\right)^2}{\Delta} = \lim_{\Delta\downarrow 0}\frac{1}{\Delta}\{\text{minimum eigenvalue of } CL_{\Delta,P}L_{\Delta,P}^*C^*\}$$
$$= \text{minimum eigenvalue of } CBPP^*B^*C^* \quad (2.25)$$

which is greater than zero because of the hypothesis, and that $\lambda_{i_{\min}}^\Delta$ is continuous with respect to Δ, the existence follows of

$$\bar{a} = \min_{\Delta\leqslant T}\frac{\lambda^{\Delta 2}}{\Delta}, \quad (2.26)$$

hence the existence of the upper bound M.

(B) It is enough to note that from (2.15) it follows that

$$(L_{\Delta,P}J_\Delta C)^2 = L_{\Delta,P}J_\Delta C. \quad (2.27)$$

That is, $L_\Delta J_\Delta C$ is indempotent and hence a projector which coincides with π as can be easily verified.

Let us go back to (2.21); first of all from (2.16), we have

$$\pi x^*(t_i) = \pi\sum_{l=0}^{i-1}\big[(I-\pi)\Gamma(\Delta)\big]^l L_{\Delta,P}J_\Delta y(t_{i-l})$$
$$= \pi L_{\Delta,P}J_\Delta Y(t_i), \quad (2.28)$$

so that

$$\|\pi x^*(t_i)\|_x \leqslant M\|y(t_i)\|_y$$
$$\leqslant M\|y\|_\infty \triangleq M_2. \quad (2.29)$$

Hence

$$4\sum_{i=1}^{N}\sum_{k=1}^{q}\alpha_k^2\int_0^{\Delta}\big[\pi x^*(t_{i-1}),(I-\Gamma(\theta))^*\varphi_k\big]_x^2\,\mathrm{d}\theta$$
$$\leqslant 4\sum_{i=1}^{N}\sum_{k=1}^{q}\alpha_k^2M_2^2\int_0^{\Delta}\|(I-\Gamma(\theta))^*\varphi_k\|_x^2\,\mathrm{d}\theta$$
$$\leqslant 4\sum_{i=1}^{N}\sum_{k=1}^{q}\alpha_k^2M_2^2\Delta\cdot\sup_{\theta\in[0,\Delta]}\|(I-\Gamma(\theta))^*\varphi_k\|_x^2$$
$$= 4TM_2^2\sum_{k=1}^{q}\alpha_k^2\sup_{\theta\in[0,\Delta]}\|(I-\Gamma(\theta))^*\varphi_k\|_x^2 \quad (2.30)$$

which is less than $\varepsilon/4$ as soon as we choose $\Delta \leqslant \Delta''$

small enough; the existence of such a Δ'' is assured by the strong continuity of the semigroup Γ^*.

We have so proved that the second term in (2.17) is less than $\varepsilon/4$ as soon as we choose $N'' \geqslant T/\Delta''$.

As far as the third term in (2.17) is concerned we have

$$4\sum_{i=1}^{N}\int_0^{\Delta}\|CL_{\theta,P}J_{\Delta}(y(t_i)-y(t_{i-1}))\|_y^2\,\mathrm{d}\theta$$
$$\leqslant 4\sum_{i=1}^{N}\sup_i\|y(t_{i-1}+\Delta)-y(t_{i-1})\|_y^2 \cdot\int_0^{\Delta}\|CL_{\theta,P}J_{\Delta}\|^2_{\mathcal{L}(\mathcal{K}_y,\mathcal{K}_y)}\,\mathrm{d}\theta. \tag{2.31}$$

Moreover it results

$$\int_0^{\Delta}\|CL_{\theta,P}J_{\Delta}\|^2_{\mathcal{L}(\mathcal{K}_y,\mathcal{K}_y)}\mathrm{d}\theta \leqslant M_3\Delta. \tag{2.32}$$

For,

$$\|CL_{\theta,P}J_{\Delta}\|_{\mathcal{L}(\mathcal{K}_y,\mathcal{K}_y)} = \sup_{\|y\|=1}\|CL_{\theta,P}J_{\Delta}y\|_y$$
$$= \sup_{\|y\|=1}\left\|\int_0^{\theta}C\Gamma(\theta-\tau)BP \cdot\sum_{k=1}^{q}\frac{[y,\varphi_k]}{\lambda_k^{\Delta}}(V_{\Delta}^*\varphi_k)(\tau)\,\mathrm{d}\tau\right\|_y$$
$$\leqslant \frac{\beta\|C\|\|BP\|}{\lambda^{\Delta}_{\min}}\sum_{k=1}^{q}\int_0^{\theta}\|(V_{\Delta}^*\varphi_k)(\tau)\|_x\,\mathrm{d}\tau$$
$$\leqslant \frac{\beta\|C\|\|BP\|}{\lambda^{\Delta}_{\min}}q\theta^{1/2} \tag{2.33}$$

where

$$\beta = \sup_{t\in[0,T]}\|\Gamma(t)\|_{\mathcal{L}(\mathcal{K}_x,\mathcal{K}_x)}.$$

By substituting (2.33) into (2.32) one has

$$\int_0^{\Delta}\|CL_{\theta,P}J_{\Delta}\|^2_{\mathcal{L}(\mathcal{K}_y,\mathcal{K}_y)}\,\mathrm{d}\theta \leqslant \frac{\beta^2\|C\|^2\|BP\|^2q^2}{2\lambda^{\Delta 2}_{\min}}\Delta^2 \leqslant M_3\Delta \tag{2.34}$$

where

$$M_3 = \frac{\beta^2\|C\|^2\|BP\|^2q^2}{2\bar{a}},$$

with $\bar{a}$ given by (2.26).

Finally, by substituting (2.34) into (2.31) we have

$$4N\sup_{|t-\tau|\leqslant\Delta}\|y(t)-y(\tau)\|_y^2M_3\Delta = 4TM_3\sup_{|t-\tau|\leqslant\Delta}\|y(t)-y(\tau)\|_y^2 \leqslant \frac{\varepsilon}{4} \tag{2.35}$$

as soon as we choose $\Delta \leqslant \Delta''$ small enough in order that

$$\|y(t)-y(\tau)\|_y \leqslant \frac{\varepsilon^{1/2}}{4T^{1/2}M_3^{1/2}}, \quad |t-\tau|\leqslant\Delta'''. \tag{2.36}$$

The existence of such a Δ''' follows by the same arguments as in (2.19). Hence for $N''' \geqslant T/\Delta'''$ (2.35) is true for any $N \geqslant N'''$.

As far as the fourth term of (2.17) is concerned:

$$4\sum_{i=1}^{N}\int_0^{\Delta}\|CL_{\theta,P}J_{\Delta}C[I-\Gamma(\Delta)]\pi x^*(t_{i-1})\|_y^2\,\mathrm{d}\theta$$
$$\leqslant 4\sum_{i=1}^{N}\|C[I-\Gamma(\Delta)]\pi x^*(t_{i-1})\|_y^2 \cdot\int_0^{\Delta}\|CL_{\theta,P}J_{\Delta}\|^2_{\mathcal{L}(\mathcal{K}_y,\mathcal{K}_y)}\,\mathrm{d}\theta$$
$$\leqslant 4M_3\sum_{i=1}^{N}\Delta\|C[I-\Gamma(\Delta)\pi]x^*(t_{i-1})\|_y^2$$
$$= 4M_3\sum_{i=1}^{N}\Delta\sum_{k=1}^{q}\alpha_k^2[(I-\Gamma(\Delta))\pi x^*(t_{i-1}),\varphi_k]_x^2$$
$$= 4M_3\sum_{i=1}^{N}\Delta\sum_{k=1}^{q}\alpha_k^2[\pi x^*(t_{i-1}), \cdot(I-\Gamma(\Delta))^*\varphi_k]_x^2$$
$$\leqslant 4M_3\sum_{i=1}^{N}\Delta\sum_{k=1}^{q}\alpha_k^2\|\pi x(t_{i-1})\|_x^2 \cdot\|(I-\Gamma(\Delta))^*\varphi_k\|_x^2$$
$$\leqslant 4M_3M_2^2T\sum_{k=1}^{q}\alpha_k^2\|(I-\Gamma(\Delta))^*\varphi_k\|_x^2 \tag{2.37}$$

which is less than $\varepsilon/4$ as soon we choose $\Delta \leqslant \Delta^{\text{iv}}$ small enough; the existence of such a Δ^{iv} is assured by the strong continuity of the semigroup Γ^*.

Hence the fourth term in (2.17) is less than $\varepsilon/4$ as soon as we choose $N^{\text{iv}} \geqslant T/\Delta^{\text{iv}}$.

By choosing

$$N_\varepsilon = \max\{N', N'', N''', N^{\text{iv}}\}, \tag{2.38}$$

(2.17) holds for any $N \geqslant N_\varepsilon$. □

Remark 1. Note that a necessary condition for (2.10) to be satisfied is that CBB^*C^* is full rank. Under this assumption a sufficient condition for the functional output ε-controllability is that the null space of C coincides with the unobservables.

Remark 2. Another condition which assures the functional output ε-controllability is that for some P satisfying (2.10) it results

$$\mathfrak{R}_{BP} \subset \mathfrak{N}(C)^{\perp}.$$

The result stated by the previous theorem can be extended to the general case of an operator $C' : \mathcal{H}_x \to \mathcal{H}_y$, with $\mathcal{H}_y$ an infinite dimensional Hilbert space, represented by

$$y = C'x = \sum_{i=1}^{\infty} \alpha_k [x, \varphi_k] \psi_k. \tag{2.39}$$

In order to state such a result we need the following preliminaries. Let $\{\Lambda_n\}_{n=1}^{\infty}$ be the sequence of projectors with finite dimensional range space, increasing strongly to identity, given by

$$\Lambda_n y = \sum_{i=1}^{n} [y, \psi_i] \psi_i \tag{2.40}$$

and denote by $\{\pi_n\}_{n=1}^{\infty}$ the family of projectors of $\mathcal{H}_x$ into the orthogonal complement of $\mathfrak{N}(\Lambda_n C')$ defined by

$$\pi_n x = \sum_{k=1}^{n} [x, \varphi_k] \psi_k, \quad n = 1, 2, \ldots. \tag{2.41}$$

Definition 2. The system (1.1), (2.39) and the corresponding forcing operator $\mathfrak{L}$, will be said to satisfy the *C-property* if the following conditions are verified:

1. $C'BB^*C'^*$ has dense image in $\mathcal{H}_y$, (2.42)
2. $(I - \pi_{\bar{n}})\mathfrak{R}_B \subset \mathcal{G}_{C'}$ for some $\bar{n} < \infty$. (2.43)

We can now state the announced result:

Theorem 2. *The system* (1.1), (2.39) *satisfies the functional-output ε-trajectory controllability if the forcing operator* $\mathfrak{L} : W_u^T \to W_x^T$, *defined by* (2.12), *satisfies the C-property.*

Proof. First of all note that for any fixed $y \in W_y^T$ and $\varepsilon > 0$ there exists an N_ε such that

$$y_N \triangleq \sum_{i=1}^{N} [y, \psi_i] \psi_i \tag{2.44}$$

has the property

$$\|y - y_N\|_{W_y^T} < \frac{\varepsilon}{2}, \quad \forall N > N_\varepsilon. \tag{2.45}$$

The proof will be achieved as soon as we prove the $\varepsilon/2$ controllability of $y_{\bar{N}}$, $\bar{N} > N_\varepsilon$. Let us choose

$$q = \max\{N_\varepsilon, \bar{n}\} \tag{2.46}$$

and let us consider the operator $C'' : \mathcal{H}_x \to \mathcal{H}_y^q$ given by $C'' = \Lambda_q C'$. It remains to show that Theorem 1 holds with reference to the finite dimensional output operator C''. For, observe that (2.42) implies

$$C''BB^*C''^* \text{ has full rank (equal to } q). \tag{2.47}$$

Moreover from (2.43) one has

$$(I - \pi_q)\mathfrak{R}_B \subset (I - \pi_{\bar{n}})\mathfrak{R}_B \subset \mathcal{G}_{C'} \subset \mathcal{G}_{C''}. \tag{2.48}$$

Hence the hypotheses of Theorem 1 are satisfied with reference to C'' which concludes the proof. □

References

[1] R.W. Brockett and M.D. Mesarovic, The reproducibility of multivariable systems, *J. Math. Anal. Appl.* 11 (1965) 548–563.

[2] M.K. Sain and J.L. Massey, Invertibility of linear time-invariant dynamic systems, *IEEE Trans. Automat. Control* 14 (1968) 141–149.

[3] H.H. Rosenbrock, On linear system theory, *Proc. IEEE* 14 (1967) 1353–1359.

[4] G. Basile and G. Marro, On the perfect output controllability of linear dynamic systems, *Ricerche di Automatica* 2 (1971) 1–10.

[5] G. Basile and G. Marro, Multi-level functional output controllability, *Int. IEEE Conf.* Oaxtepec, Morelos, Mexico (January 1971).

[6] G. Basile and F. Hamano, On the smoothness of the output trajectories for a linear dynamic system, *IEEE Trans. Automat. Control* 27 (1982) 196–198.

[7] V. Iliris and M. Megan, Controllability of linear systems

and a boundary value problem, *Aequationes Mathematicae* 16 (1977) 81–91.

[8] A. Germani and S. Monaco, Some results on the controllability of perturbed linear systems on Hilbert spaces, *Syst. Control Lett.* 1 (2) (1981) 140–146.

[9] H.O. Fattorini, Some remarks on complete controllability, *SIAM J. Control* 4 (4) (1966) 686–694.

[10] A.V. Balakrishnan, *Applied Functional Analysis* (Springer, Berlin–New York, 1976).

[11] P.R. Halmos, *A Hilbert Space Problem Book* (Van Nostrand, New York, 1967).