# Some results on the controllability of perturbed linear systems on Hilbert spaces

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This paper studies the controllability of the trajectories for the class of linear perturbed systems whose state space is a separable Hilbert space.

In particular it is proved that under quite general assumptions the controllability of the linear part of the system (unperturbed system) assures the controllability of the whole system.

*Keywords*: Controllability, Infinite dimensional systems, Perturbed linear systems.

## 1. Introduction

This paper deals with the problem of studying the controllability of the trajectories for a class of nonlinear systems whose state, input and output spaces are separable Hilbert spaces.

More precisely we will consider linear perturbed control systems described by

$$\dot{x}(t) = Ax(t) + Bu(t) + g(x(t), u(t)), \qquad x(0) = \hat{x}, \tag{1.1}$$

where $x(t) \in \mathcal{H}_x$ and $u(t) \in \mathcal{H}_u$, $\mathcal{H}_x$ and $\mathcal{H}_u$ being separable real Hilbert spaces with $[\cdot,\cdot]_x$, $[\cdot,\cdot]_u$ and $|\cdot|_x$, $|\cdot|_u$ the inner products and norms respectively. Moreover, $A$ is the infinitesimal generator of a strongly continuous semigroup of operators $\{\Gamma(t),\ t \geq 0\}$; $B: \mathcal{H}_u \to \mathcal{H}_x$ is a bounded linear operator; $g: \mathcal{H}_u \times \mathcal{H}_x \to \mathcal{H}_x$ is a perturbation term which satisfies the Lipschitz conditions

$$|g(x_1, u) - g(x_2, u)|_x \leq K_1(|u|_u)|x_1 - x_2|_x, \tag{1.2}$$

$$|g(x, u_1) - g(x, u_2)|_x \leq K_2(|x|_x)|u_1 - u_2|_u, \tag{1.3}$$

where $K_1(|u|_u) \leq K|u|_u$, $K_2(|x|_x) \leq K|x|_x$ and $g(0,u) = g(x,0) = 0$, for any choice of $u \in \mathcal{H}_u$ and $x \in \mathcal{H}_x$.

We will assume the input and the state to belong to the real separable Hilbert spaces $W_u^T \triangleq L_2[(0,T); \mathcal{H}_u]$ and $W_x^T \triangleq L_2[(0,T); \mathcal{H}_x]$ respectively. In the following we will denote by $\|\cdot\|_{W_x^t}$ and $\|\cdot\|_{W_u^t}$ the norms in $W_x^t$ and $W_u^t$ respectively.

It is well known [1] that, under the above assumptions, the equation (1.1) has to be interpreted as a representation of the following integral equation:

$$[x(t), a] = [\hat{x}, a] + \int_0^t [x(s), A^*a]\,\mathrm{d}s + \int_0^t [Bu(s), a]\,\mathrm{d}s + \int_0^t [g(x(s), u(s)), a]\,\mathrm{d}s \tag{1.4}$$

for $t \in [0,T]$ and any $a \in \mathcal{D}(A^*)$ (the domain of the adjoint $A^*$ of $A$).

Arguing as in [2] it is easy to show that any solution of (1.4) in $W_x^T$ is a solution of the equation

$$x = x_0 + Lu + G(x, u) \tag{1.5}$$

 

and vice versa. We assume that:

– $x_0(t)=\Gamma(t)\hat{x}$, (1.6)

– $L: W_u^T \to W_x^T$ is defined by

$$y=Lu; \qquad y(t)=\int_0^t \Gamma(t-\tau)Bu(\tau)\,\mathrm{d}\tau, \tag{1.7}$$

– $G: W_u^T \times W_x^T \to W_x^T$ is defined by

$$z=G(x,u); \qquad z(t)=\int_0^t \Gamma(t-\tau)g(x(\tau),u(\tau))\,\mathrm{d}\tau. \tag{1.8}$$

In the sequel we will study controllability of a system described by (1.5). Section 2 concernes with the existence and uniqueness of the solution for the equation (1.5). In Section 3 we state some preliminary results about compact linear operators. On these bases, we give our main result on the preservation of the controllability property once the perturbation term is added to state equations of a linear system (equation (1.5)), provided that boundedness conditions on the perturbation term are verified.

The results stated, as will be clarified in Section 3, concern mainly trajectory controllability.

In the opinion of the authors, trajectory controllability, which seems to have a central role in many control problems, has not been sufficiently investigated even for finite dimensional linear systems. Moreover, such a kind of controllability appears to be more adequate when a functional analysis approach is used.

## 2. Preliminary results

As far as existence and uniqueness of the solution of (1.5) is concerned, we can state the following result:

**Theorem 1**. *The integral version* (1.4) *of the system* (1.1) *under the assumptions* (1.2), (1.3) *has a unique solution* $x \in W_x^T$ *for every* $u \in W_u^T$.

**Proof**. As we have observed, it is sufficient to prove that (1.5) has a unique solution in $W_x^T$. At this end let us consider the following chain of equations:

$$x_{n+1}=x_0+Lu+G(x_n,u). \tag{2.1}$$

First we will show that $\{x_i\}$ is a bounded sequence in $W_x^T$.

$$\begin{aligned}\|x_{n+1}\|_{W_x^T}^2 &\leqslant 2\|x_0+Lu\|_{W_x^T}^2+2\|G(x_n,u)\|_{W_x^T}^2\\ &\leqslant 2\|x_0+Lu\|_{W_x^T}^2+2\int_0^T\left|\int_0^t\Gamma(t-\tau)g(x_n(\tau),u(\tau))\,\mathrm{d}\tau\right|_x^2\mathrm{d}t\\ &\leqslant 2\|x_0+Lu\|_{W_x^T}^2+2M^2K^2\int_0^T\left[\int_0^t|u(\tau)|_u|x_n(\tau)|_x\,\mathrm{d}\tau\right]^2\mathrm{d}t\\ &\leqslant 2\|x_0+Lu\|_{W_x^T}^2+2M^2K^2\|u\|_{W_u^T}^2\int_0^T\|x_n\|_{W_x^T}^2\,\mathrm{d}t\end{aligned} \tag{2.2}$$

where $M=\sup_{t\in[0,T]}|\Gamma(t)|_{L(\mathcal{X}_x,\mathcal{X}_x)}$ and the Schwartz inequality was used.

By iterating (2.2) we get

$$\|x_{n+1}\|^2_{W_x^T} \leqslant 2\sum_{i=0}^{n} \frac{\left(2M^2K^2\|u\|^2_{W_u^T}T\right)^i}{i!}\|x_0 + Lu\|^2_{W_x^T}$$
$$+\left(2M^2K^2\|u\|^2_{W_u^T}\right)^{n+1}\int_0^T\int_0^{t_1}\cdots\int_0^{t_n}\|x_0\|^2_{W_x^{t_{n-1}}}\,\mathrm{d}t_{n+1}\cdots\mathrm{d}t_1$$
$$\leqslant 2\|x_0 + Lu\|^2_{W_x^T}\sum_{i=0}^{n}\frac{\left(2M^2K^2\|u\|^2_{W_u^T}T\right)^i}{i!} + \left(2M^2K^2\|u\|^2_{W_u^T}\right)^{n+1}\|x_0\|^2_{W_x^T}\frac{T^{n+1}}{(n+1)!}$$
$$\leqslant\left[2\|x_0 + Lu\|^2_{W_x^T} + \|x_0\|^2_{W_x^T}\right]\sum_{i=0}^{n+1}\frac{\left(2MK\|u\|^2_{W_u^T}T\right)^i}{i!}$$
$$\leqslant 2\left(\|x_0 + Lu\|^2_{W_x^T} + \|x_0\|^2_{W_x^T}\right)\exp\left(2MK\|u\|^2_{W_u^T}T\right)$$
$$\triangleq \alpha\left(\|x_0\|_{W_x^T}, \|u\|_{W_u^T}, T\right) < \infty. \tag{2.3}$$

Next we will show that $x_i$ is a Cauchy sequence.

$$\|x_{n+p} - x_n\|^2_{W_x^T} = \|G(x_{n+p-1},u) - G(x_{n-1},u)\|^2_{W_x^T}$$
$$\leqslant\int_0^T\left|\int_0^{t_1}\Gamma(t-\tau)\left[g(x_{n+p-1}(\tau),u(\tau)) - g(x_{n-1}(\tau),u(\tau))\right]\mathrm{d}\tau\right|_x \mathrm{d}t_1$$
$$\leqslant M^2K^2\|u\|^2_{W_u^T}\int_0^T\|x_{n+p-1} - x_{n-1}\|^2_{W_x^{t_1}}\,\mathrm{d}t_1. \tag{2.4}$$

By iterating (2.4) and by (2.3) we have

$$\|x_{n+p-1} - x_{n-1}\|^2_{W_x^T} \leqslant \left(M^2K^2\|u\|^2_{W_u^T}\right)^n\frac{T^n}{n!}\|x_p - x_0\|^2_{W_x^T}$$
$$\leqslant\frac{\left(M^2K^2\|u\|^2_{W_u^T}T\right)^n}{n!}\left(2\|x_0\|^2_{W_x^T} + 2\alpha\left(\|x_0\|_{W_x^T},\|u\|_{W_u^T}\right)\right), \tag{2.5}$$

which goes to zero for $n$ going to infinity. Now, because of the completeness of $W_x^T$, the limit is well defined:

$$x = \lim_{i\to\infty} x_i \tag{2.6}$$

which is the announced solution of (1.4) as can be easily verified taking into account the continuity of $G$ with respect to $x$.

It remains to show the uniqueness. At this end let $x$ and $y$ be two different solutions of (1.4); then we have

$$\|x-y\|^2_{W_x^T} = \|G(x,u) - G(y,u)\|^2_{W_x^T} \leqslant K^2M^2\|u\|^2_{W_u^T}\int_0^T\|x-y\|^2_{W_x^t}\,\mathrm{d}t \tag{2.7}$$

which implies by the Gronwall inequality $\|x-y\|^2_{W_x^T} = 0$; i.e. the uniqueness of the solution. □

**Remark 1**. Note that the above theorem remains true if $K_1(|u|_u) \leqslant N$.

Before studying the controllability of the system (1.5), let us observe that the controllability problem for linear systems can be reduced to the problem of 'controllability' of a linear operator.

To be more precise, let us introduce the following definition:

**Definition 1.** A map $T:\mathcal{H}_1 \to \mathcal{H}_2$, $\mathcal{H}_1$ and $\mathcal{H}_2$ separable Hilbert spaces, is said to be *$\epsilon$-controllable in* $y \in \mathcal{H}_2$ if for any $\epsilon > 0$ there exists $z_{\epsilon,y} \in \mathcal{H}_1$ such that

$$\|Tz_{\epsilon,y} - y\|_{\mathcal{H}_2} < \epsilon, \tag{2.8}$$

while it is said to be *$\epsilon$-controllable* if (2.8) holds for any $y$ in $\mathcal{H}_2$.

Note that when $T:L_2[(0,T),\mathcal{H}_u] \to \mathcal{H}_x$ is the input-state map of the unperturbed linear system associated to (1.5), Definition 1 restitutes the well-known $\epsilon$-controllability property [4,5]. Moreover, when $\mathcal{H}_u = \mathbb{R}^p$ and $\mathcal{H}_x = \mathbb{R}^n$ $\epsilon$-controllability is equivalent to exact state controllability. When $T:L_2[(0,T);\mathcal{H}_u] \to L_2[(0,T);\mathcal{H}_x]$, even in the finite dimensional case, it is necessary to consider the $\epsilon$-controllability property. In the sequel when the range space of the operator $T$ is $L_2[(0,T);\mathcal{H}_x]$ we will refer to the controllability property of Definition 1 as *$\epsilon$-trajectory controllability.*

The following result, quite interesting by itself, will be useful for the analysis performed in the next section.

**Lemma 1**. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real separable Hilbert spaces and $T:\mathcal{H}_1 \to \mathcal{H}_2$ a compact, $\epsilon$-controllable, linear operator. Then given $\epsilon > 0$, for any compact set $K \subset \mathcal{H}_2$ there exists a continuous linear map $\mathcal{J}_\epsilon:\mathcal{H}_2 \to \mathcal{H}_1$ such that*

$$\|T \circ \mathcal{J}_\epsilon x - x\|_{\mathcal{H}_2} < \epsilon \quad \forall x \in K. \tag{2.9}$$

**Proof.** First of all let us consider the polar decomposition of $T = L \cdot V$, where $L:\mathcal{H}_2 \to \mathcal{H}_2$ is a compact self-adjoint operator and $V:\mathcal{H}_1 \to \mathcal{H}_2$ is a unitary operator ($VV^* = V^*V = I$). Moreover, we will denote by $\{\phi_i\}$, $i = 1,2,\ldots$ the eigenvectors of the operator $TT^*$, which constitute a basis for $\mathcal{H}_2$. For any $x \in \mathcal{H}_2$ and for any given $\epsilon > 0$ there exists a $\delta_{\epsilon,x}$ such that

$$z_\epsilon(x) \triangleq \sum_{i=1}^{\infty} \frac{[x,\phi_i]_{\mathcal{H}_2}}{\sqrt{\lambda_i} + \delta_{\epsilon,x}} V^*\phi_i \tag{2.10}$$

verifies the $\epsilon$-controllability property in $x$. For,

$$\begin{aligned}
\|Tz_\epsilon(x) - x\|^2_{\mathcal{H}_2} &= \left\| \sum_{i=1}^{\infty} \frac{[x,\phi_i]_{\mathcal{H}_2}}{\sqrt{\lambda_i} + \delta_{\epsilon,x}} TV^*\phi_i - \sum_{i=1}^{\infty} [x,\phi_i]_{\mathcal{H}_2}\phi_i \right\|^2_{\mathcal{H}_2} \\
&= \sum_{i=1}^{\infty} [x,\phi_i]^2_{\mathcal{H}_2} \left( \frac{\delta_{\epsilon,x}}{\sqrt{\lambda_i} + \delta_{\epsilon,x}} \right)^2 \\
&= \sum_{i=1}^{N_\epsilon} [x,\phi_i]^2_{\mathcal{H}_2} \left( \frac{\delta_{\epsilon,x}}{\sqrt{\lambda_i} + \delta_{\epsilon,x}} \right)^2 + \sum_{i=N_\epsilon+1}^{\infty} [x,\phi_i]^2_{\mathcal{H}_2} \left( \frac{\delta_{\epsilon,x}}{\sqrt{\lambda_i} + \delta_{\epsilon,x}} \right)^2 \\
&\leqslant \frac{\|x\|^2 N_\epsilon}{\sqrt{\lambda_{\min}}} \delta^2_{\epsilon,x} + \sum_{i=N_\epsilon+1}^{\infty} [x,\phi_i]^2_{\mathcal{H}_2}
\end{aligned} \tag{2.11}$$

where $\lambda_{\min} = \min\{\lambda_i, i = 1,2,\ldots,N_\epsilon\}$. Choose now $N_\epsilon$ large enough so that the sum is smaller than $\frac{1}{2}\epsilon$ and

$$\delta^2_{\epsilon,x} \leqslant \frac{\epsilon\sqrt{\lambda_{\min}}}{2N_\epsilon \|x\|^2_{\mathcal{H}_2}}. \tag{2.12}$$

Let $K$ be any compact set in $\mathcal{H}_2$; then there exists a finite cover of $K$ by $\epsilon$-radius open spheres:

$$K \subset \sum_{i=1}^{N} S(x_i, \tfrac{1}{3}\epsilon). \tag{2.13}$$

Let us now define

$$\delta \triangleq \min\{\delta_{\epsilon/3,x_i}, i=1,2,\dots,N\}; \tag{2.14}$$

then the map $\mathcal{J}_\epsilon : \mathcal{H}_2 \to \mathcal{H}_1$ defined by

$$\mathcal{J}_\epsilon x = \sum_{i=1}^{\infty} \frac{[x,\phi_i]_{\mathcal{H}_2}}{\sqrt{\lambda_i}+\delta} V^*\phi_i \tag{2.15}$$

is obviously linear. Moreover, for $x \in K$ we have that there exists $\bar{i}$ such that

$$x \in S\left(x_{\bar{i}}, \tfrac{1}{3}\epsilon\right) \tag{2.16}$$

and

$$\|T\circ\mathcal{J}_\epsilon x - x\|_{\mathcal{H}_2} = \|T\circ\mathcal{J}_\epsilon x - T\circ\mathcal{J}_\epsilon x_{\bar{i}}\|_{\mathcal{H}_2} + \|T\circ\mathcal{J}_\epsilon x_{\bar{i}} - x_i\|_{\mathcal{H}_2} + \|x_{\bar{i}} - x\| \leqslant \epsilon. \tag{2.17}$$

Finally observe that from (2.15)

$$\|\mathcal{J}_\epsilon x\|_{\mathcal{H}_1} < \frac{1}{\delta}\|x\|_{\mathcal{H}_2}, \tag{2.18}$$

which assures the continuity of $\mathcal{J}_\epsilon$. □

## 3. Controllability of the perturbed system

We are now in the condition to study the controllability for the system (1.5). More precisely it will be found that, under suitable conditions on the perturbation term in (1.5), if a trajectory is $\epsilon$-controllable for the unperturbed system the same is true for the whole system.

To state our main result we need the following lemma:

**Lemma 2.** *The Volterra operator $L_p : W_x^T \to W_x^T$ defined by*

$$y = L_p z; \qquad y(t) = \int_0^t \Gamma(t-\tau) z(\tau)\,\mathrm{d}\tau \tag{3.1}$$

*is compact provided that $\{\Gamma(t),\, t>0\}$ is a strongly continuous semigroup of compact operators.*

**Proof.** Let $L_\epsilon : W_x^T \to W_x^T$, $\epsilon \geqslant 0$, be the linear operator defined by

$$y = L_\epsilon z; \qquad y(t) = \int_0^{t-\epsilon} \Gamma(t-\tau) z(\tau)\,\mathrm{d}\tau \tag{3.2}$$

where $\Gamma(t)$ and $z(t)$ are assumed zero for $t<0$. We start by proving that $L_\epsilon$ as in (3.2) is compact for any $\epsilon>0$. For, we have the estimate

$$\|(L_\epsilon z)(t)\|_{H_x} \leqslant M\sqrt{T}\,\|z\|_{W_x^T}; \quad \epsilon \geqslant 0, \tag{3.3}$$

and, for any $h>0$, $t\in[0,T]$ and $z \in W_x^T$

$$\|(L_\epsilon z)(t+h) - (L_\epsilon z)(t)\|_{H_x} \leqslant M\|z\|_{W_x^T}\sqrt{h} + \sqrt{T}\sup_{t\in[\epsilon,T]}\|\Gamma(t+h)-\Gamma(t)\|_{L(H_x^T,H_x^T)}\|z\|_{W_x^T}. \tag{3.4}$$

Moreover, by the Lax theorem which assures the uniform continuity of $\Gamma(t)$ in $[\epsilon, T]$ [8, Th. 10.2.2], (3.4) goes to zero whenever $h \downarrow 0$. Hence the image of the unit ball in $W_x^T$ under the action of $L_\epsilon$, $\epsilon>0$, is an equicontinuous family of functions (they vanish in $[0,\epsilon]$). Moreover $L_\epsilon$ may be easily expressed as

$$(L_\epsilon z)(t) = \Gamma(\epsilon)(L_p z)(t-\epsilon) \tag{3.5}$$

which shows that

$$\{(L_\epsilon z)(t);\ \|z\|_{W_x^T} \leqslant 1\} = \Gamma(\epsilon)\{(L_0 z)(t-\epsilon);\ \|z\|_{W_x^T} \leqslant 1\}. \tag{3.6}$$

Since the set $\{(L_0 z)(t-\epsilon);\ \|z\|_{W_x^T} \leqslant 1\}$ is, by (3.3), bounded and $\Gamma(\epsilon)$ compact, then the set in the left-hand side of (3.6) is precompact. This means, by the Arzela–Ascoli theorem, that $L_\epsilon$, $\epsilon > 0$, is a compact operator treated as an operator from $W_x^T$ into $C([0,T];\ H_x)$. □

**Theorem 2.** *Any trajectory $\bar{x} \in W_x^T$ is $\epsilon$-controllable for the system* (1.5) *if:*

(i) *the unperturbed part of the system* (1.5) *is $\epsilon$-trajectory-controllable,*

(ii) $K_1(|u|_u) \leqslant N$, $\forall u \in \mathcal{H}_u$,

(iii) $\{\Gamma(t),\ t > 0\}$ *is a compact semigroup.*

**Proof.** Let $L_p\colon W_x^T \to W_x^T$ be a linear operator defined by

$$y = L_p z; \qquad y(t) = \int_0^t \Gamma(t-\tau) z(\tau)\,\mathrm{d}\tau \tag{3.7}$$

which is compact because of (iii) and Lemma 2. Moreover, let us introduce the nonlinear map $h_{\bar{x}}$: $W_x^T \to W_x^T$ defined by

$$y = h_{\bar{x}}(u); \qquad y(t) = g(\bar{x}(t), u(t)). \tag{3.8}$$

Because of (ii) we have

$$\|h_{\bar{x}}(u)\|^2_{W_x^T} = \int_0^T |g(\bar{x}(t), u(t))|_x^2\,\mathrm{d}t \leqslant N^2 \|\bar{x}\|^2_{W_x^T}. \tag{3.9}$$

Hence we have

$$G(\bar{x}, u) = L_p \circ h_{\bar{x}}(u). \tag{3.10}$$

It results that the set

$$\left\{x\colon x = \bar{x} - G(\bar{x}, u),\ u \in W_u^T\right\} \tag{3.11}$$

is contained in a compact set, namely $K_{\bar{x}}$, in $W_x^T$, because $L_p$ is a compact linear operator and $h_{\bar{x}}(u)$ is contained in a closed sphere with radius $N\|\bar{x}\|_{W_x^T}$. It follows from Lemma 1 that for any given $\epsilon' > 0$ there exists a $\mathcal{J}_{\epsilon'}$ such that

$$\|L\mathcal{J}_{\epsilon'} x - x\|_{W_x^T} < \epsilon' \quad \forall x \in K_{\bar{x}} \tag{3.12}$$

where $\mathcal{J}_{\epsilon'}$ and $L$ are defined in (2.15) and (1.7) respectively.

In order to show that for any $\epsilon > 0$ there exists $\bar{u}$ such that

$$\|\bar{x} - x_{\bar{u}}\|_{W_x^T} < \epsilon \tag{3.13}$$

where $x_{\bar{u}}$ is the solution of (1.5) corresponding to $\bar{u}$, we introduce the following input sequence:

$$u_0 = 0, \qquad u_{n+1} = \mathcal{J}_{\epsilon'}(\bar{x} - G(\bar{x}, u_n)). \tag{3.14}$$

This is a Cauchy sequence, for

$$\begin{aligned}
\|u_{n+p} - u_n\|^2_{W_u^T} &= \|-\mathcal{J}_{\epsilon'} G(\bar{x}, u_{n+p-1}) + \mathcal{J}_{\epsilon'} G(\bar{x}, u_{n-1})\|^2_{W_u^T} \\
&\leqslant \left(\frac{1}{\delta}\right)^2 \|G(\bar{x}, u_{n+p-1}) - G(\bar{x}, u_{n-1})\|^2_{W_x^T} \\
&= \left(\frac{1}{\delta}\right)^2 \int_0^T \left| \int_0^t \Gamma(t-\tau)\left[g(\bar{x}(\tau), u_{n+p-1}(\tau)) - g(\bar{x}(\tau), u_{n-1}(\tau))\right] \mathrm{d}\tau \right|_x^2 \mathrm{d}t \\
&\leqslant \left(\frac{1}{\delta}\right)^2 M^2 K^2 \int_0^T \left[\int_0^t |\bar{x}(\tau)|_x |u_{n+p-1}(\tau) - u_{n-1}(\tau)|_u\,\mathrm{d}\tau\right]^2 \mathrm{d}t \\
&\leqslant \delta^{-2} M^2 K^2 \|\bar{x}\|^2_{W_x^T} \int_0^T \|u_{n+p-1} - u_{n-1}\|^2_{W_u^{t_1}}\,\mathrm{d}t_1.
\end{aligned} \tag{3.15}$$

By iterating (3.15) we obtain that

$$\|u_{n+p}-u_n\|^2_{W_u^T} \leq \left(\delta^{-2}M^2K^2\|\bar{x}\|^2_{W_x^T}\right)^n \int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}}\|u_p\|^2_{W_u^{t_n}}\,\mathrm{d}t_n\cdots\mathrm{d}t_1$$
$$\leq \left(\delta^{-2}M^2K^2\|\bar{x}\|^2_{W_x^T}\right)^n \frac{T^n}{n!}\|u_p\|^2_{W_u^T}. \tag{3.16}$$

Moreover,

$$\|u_p\|^2 \leq \delta^{-2}\left(\|\bar{x}\|^2_{W_x^T}+\|G(\bar{x},u_{p-1})\|^2_{W_x^T}\right) \leq 2\delta^{-2}\|\bar{x}\|^2_{W_x^T}(1+M^2N^2T^2)<\infty. \tag{3.17}$$

(3.16) and (3.17) imply that the sequence defined by (3.14) is a Cauchy sequence. Let us now define $\bar{u}$ by

$$\bar{u}=\lim_{n\to\infty}u_n. \tag{3.18}$$

It follows that

$$\|\bar{x}-G(\bar{x},\bar{u})-L\mathcal{J}_{\epsilon'}(\bar{x}-G(\bar{x},\bar{u}))\|_{W_x^T} = \|\bar{x}-G(\bar{x},\bar{u})-L\bar{u}\|_{W_x^T}<\epsilon'. \tag{3.19}$$

(3.19) can be used to show that (3.13) is true, for

$$\|\bar{x}-x_{\bar{u}}\|_{W_x^T} = \|\bar{x}-L\bar{u}-G(x_{\bar{u}},\bar{u})-G(\bar{x},\bar{u})+G(\bar{x},\bar{u})\|_{W_x^T}$$
$$\leq \epsilon'+\|G(x_{\bar{u}},\bar{u})-G(\bar{x},\bar{u})\|_{W_x^T}$$
$$\leq \epsilon'+\left(\int_0^T\left|\int_0^t\Gamma(t-\tau)\left[g(x_{\bar{u}}(\tau),\bar{u}(\tau))-g(\bar{x}(\tau),\bar{u}(\tau))\right]\mathrm{d}\tau\right|_X^2\mathrm{d}t\right)^{1/2}$$
$$\leq \epsilon'+M\left(\int_0^T\left[\int_0^t|g(x_{\bar{u}}(\tau))-g(\bar{x}(\tau),\bar{u}(\tau))|_X\,\mathrm{d}\tau\right]^2\mathrm{d}t\right)^{1/2}$$
$$\leq \epsilon'+MNT^{1/2}\left(\int_0^T\|\bar{x}-x_u\|^2_{W_x^t}\,\mathrm{d}t\right)^{1/2}. \tag{3.20}$$

Squaring and using the Gronwall inequality one obtains

$$\|\bar{x}-x_{\bar{u}}\|_{W_x^T} \leq \sqrt{2}\left(1+2M^2N^2T^2\,\mathrm{e}^{2M^2N^2T^2}\right)^{1/2}\epsilon' \leq \sqrt{2}\ \mathrm{e}^{2M^2N^2T^2}\epsilon'=\epsilon. \tag{3.21}$$

Choosing $\epsilon'=\mathrm{e}^{-2M^2N^2T^2}\epsilon/\sqrt{2}$, (3.13) holds true. □

**Remark 2.** Note that the theorem is also true substituting condition (iii) by
(iii)′ $h_{\bar{x}}(W_u^T)\subset K$, $K$ a compact set.
This is the case when, for instance, $h_{\bar{x}}(u)=D_{\bar{x}}h'_{\bar{x}}(u)$, $h'_{\bar{x}}$ defined as in (3.8) and $D_{\bar{x}}$ a compact linear operator.

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