RICERCHE DI AUTOMATICA
Vol. 9, No. 2, December 1978
Printed in Italy

AN INTERPRETATIVE MODEL FOR THE SCHOOL SERVICE DEMAND

C. Gori-Giorgi, S. Monaco and G. G. Rizzo

Abstract. In this paper we study the links between two phenomena: the increase in the number of school students and the decrease in the number of land workers.

After an informal description, a model is developed consisting of two major subsystems: a demographic subsystem, describing the school service demand of the population, and a school subsystem, on which some external variables, such as industrial investments, act as inputs.

The proposed model is a dynamical, nonlinear discrete time system and is found to be in good agreement with available data for the Italian Society.

1. Introduction.

In this paper we shall deal with two important changes that characterized the Italian society in the last 30 years: the considerable increase in the number of school students (fig. 1 *a*) and the decrease in the number of land-workers (fig. 1 *b*).

It is natural to think that these two phenomena are connected, at least, for the following reasons:

i) country people are less interested than town people in keeping their sons to school; therefore a decrease in the number of land-workers implies an increase in the number of school students;

ii) students of higher schools are not interested in agriculture work; therefore an increase in the number of students implies a decrease in the number of land-workers.

Manuscript received September 4, 1979.

C. Gori-Giorgi and S. Monaco are with the Istituto di Automatica, University of Rome, via Eudossiana 18, 00184 Rome, Italy. G. G. Rizzo is with the Istituto di Pianificazione Territoriale, University of Rome, via Cassia 32, 00191 Rome, Italy.

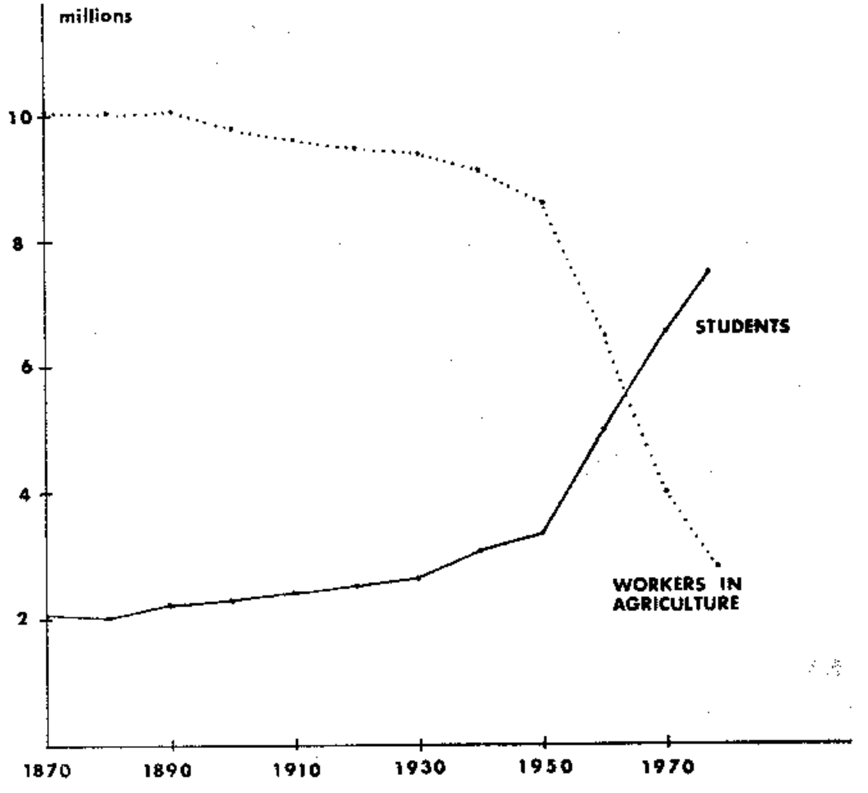
millions
10
8
6
4
2
1870
1890
1910
1930
1950
1970
STUDENTS
WORKERS IN
AGRICULTURE

fig. 1 *a*

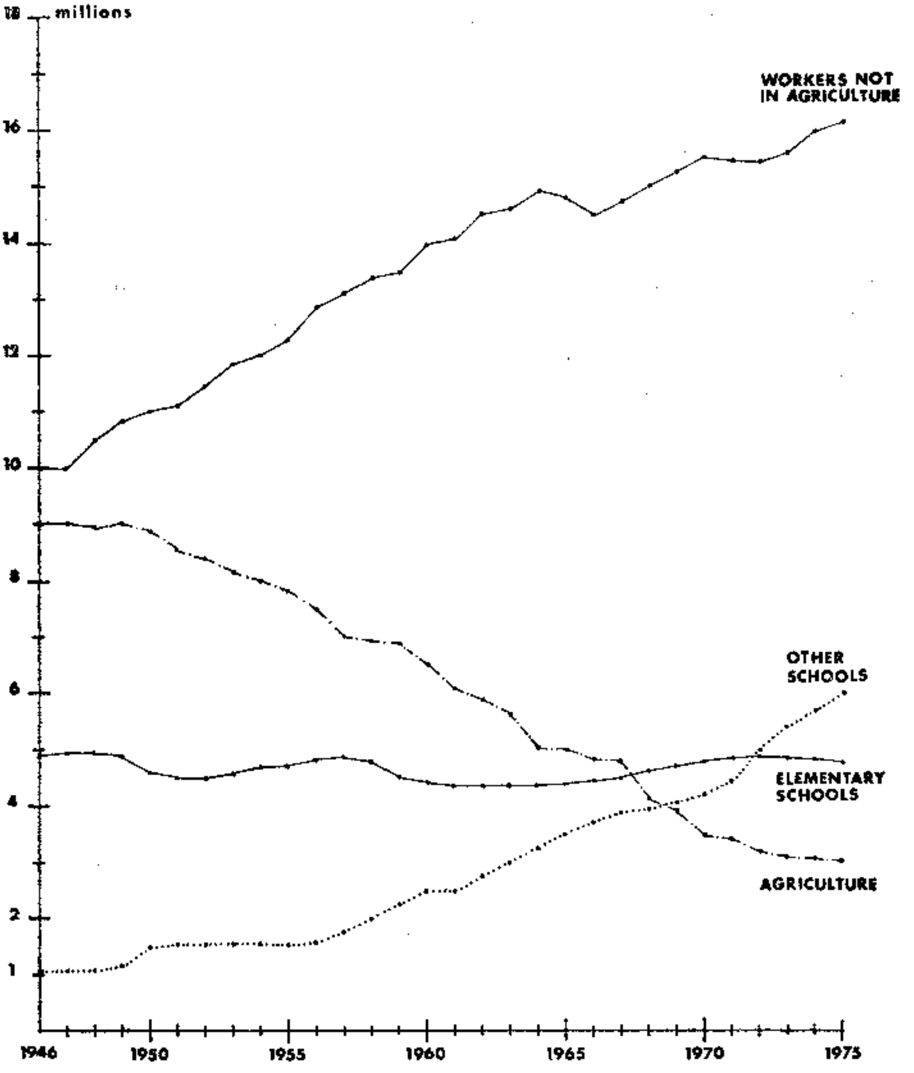
millions
18
16
14
12
10
8
6
4
2
1
1946
1950
1955
1960
1965
1970
1975
WORKERS NOT IN AGRICULTURE
OTHER SCHOOLS
ELEMENTARY SCHOOLS
AGRICULTURE

fig. 1 *b*

In order to interpret and test the role of the school with respect to these two hypotheses, we shall develop a mathematical model. The interpretative purpose requires a model where some simplifying assumptions seem to be well justified.

The sequel of this paper is divided into 5 sections:

Section 1 is the introduction; Section 2 gives a general description of the model in an informal manner; Section 3 states the basic equations representing the stocks and flows of population; Section 4 gives the whole model in the form of a dynamical discrete time nonlinear system; and at last, Section 5 considers the application of the model to Italy considering some further hypotheses to overcome the difficulties in collecting data. The results are so in agreement with reality that they give a first validation to the model.

Before ending this section, we give some information on the Italian School System. Roughly speaking, the Italian School System consists of 4 cycles of studies that are:

i) Elementary (or Primary) School - from the age of 6 up to 10;

ii) Secondary (or Junior High) School - from the age of 11 up to 13;

iii) Senior High School, with its distinction in Grammar School, Art School, Technical College, etc. - from the age of 14 up to 18;

iv) University, which lasts at least 4 to 6 years according to the faculty chosen - from the age of 19.

At the end of each cycle of studies, students have to take an examination and if they pass it they get the relative degree.

The Elementary (or Primary) School is compulsory by law since 1859, while the Secondary (or Junior High) School became compulsory only in 1962.

2. A look at the model.

In this section we give a complete but informal description of our model. A careful reading of this section will help the reader, who is not interested in mathematical developments, to better understand the contents of Sections 3 and 4.

First of all, we must point out that in order to describe the reduction of manpower in agriculture and the increase in the number of students, we have to distinguish the population in three groups, at least; however we think it useful to make one more distinction between unemployed and employed population. This distinction is selfexplanatory and is useful to understand the behaviour of the population. Thus we divide the population into four groups: land-workers, workers other than agricultural, students and others.

From year to year people migrate from one group to another thus changing their position. The migration due to school has special characters, because people joining school are usually sent back to other groups only after an almost fixed number of years. Therefore, school acts as a finite delay.

Furthermore, there is no doubt that school deeply biases the population passing through, and addresses it to one kind of work rather than to another.

This is depicted in fig. 2, where for simplicity we suppose the existence of only two levels in the school system; we left out the line from the second level to the work in agriculture just to illustrate the aforesaid fact.

The two roles of the school in our model, i. e. delaying and sharing/shunting, are also put in evidence in fig. 2.

The keyrole in the model is played by the flows between blocks; these flows depend on some control variables, or inputs, as represented by dashed lines in fig. 2. The structure of the model is as follows: the first input, i. e. the industrial investment at each year, determines the changes in the number of workers in industry and in agriculture; an increase in investments implying a decrease in agricultural employment.

The link between these variables is dynamic, that is the change in the number of workers in agriculture at time t depends on the investments at time t, $t-1$, $t-2$, and so on.

The agricultural work does not require high school degrees and low aged people can easily set to work in agriculture; moreover, in small towns or in rural villages the school building may be very far. Then we assume that students belonging to farm workers families have less chances to pass from one school cycle to the following than other students. Thus, the probability of passing from one school cycle to the next one is determined by the structure of working population.

Another input that influences this is the difficulty of the school, and a third input appearing in the model is the investment of the Government in agriculture.

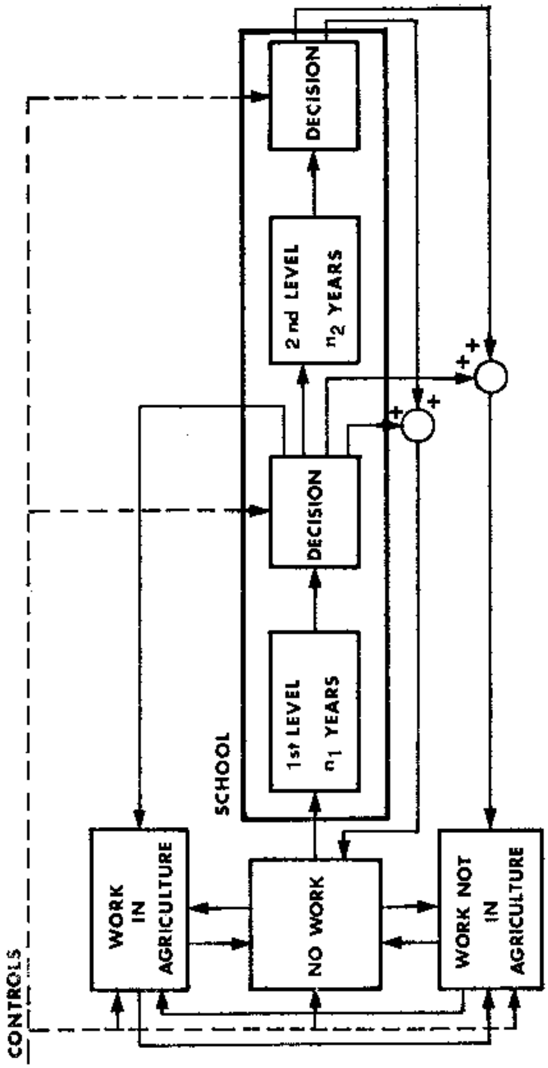
CONTROLS
SCHOOL
1st LEVEL
n_1 YEARS
DECISION
2nd LEVEL
n_2 YEARS
DECISION
WORK IN AGRICULTURE
NO WORK
WORK NOT IN AGRICULTURE

fig. 2

Going on with our general description, we note that the school is strongly linked to the age, and that a good description of the school can be given only if the population is divided into age groups. Hence each of the parts of population is divided into suitable age groups; people pass from an age group to the other according to survival coefficients and are generated into the first group according to birth-rate coefficients. Thus the blocks in fig. 2 split into more blocks as in fig. 3 and a further splitting arises from the distinction between male and female population.

As far as the control variables are constant, the model becomes similar to a modified version of the Leslie model with migration [1, 2] and this gives confidence in it. However, when the control variables are not constant with time, the model becomes a nonlinear system. We shall restrict ourselves to the case of small variations of the controls, thus assuming a linear dependence and obtaining a system which is linear in the input but not in the state.

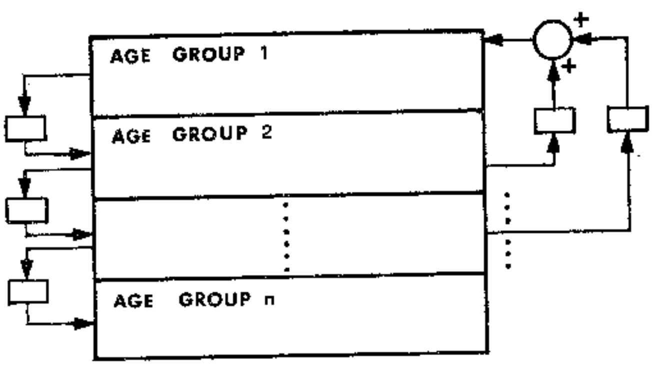

fig. 3

3. The demographic and school subsystems.

From the ideas given in the previous sections, we assume that at each time t a population P has been unambiguously fixed and divided into the following parts:

i) P_v: elements of P working in agriculture, whose number will be denoted by v;

ii) P_w: students, from Elementary (or Primary) School up to University, whose number will be denoted by w;

iii) P_y: elements of P working not in agriculture, whose number will be denoted by y;

iv) P_z: elements of P not belonging to i), ii) or iii), that is neither working nor studying population; we shall call this unemployed population and denote its number by z.

We must point out that the above defined sets are assumed to be disjointed; this will require some care when collecting data, especially, as per our experience, for w.

We further assume that each of the above parts is divided into age groups according to Table 1. Our choice of the age groups is mainly linked to school and labour laws in force in Italy, as depicted in Section 1. The model can, of course, be used with other types of age groups.

Finally, from a look at school and manpower data available, we have to make a distinction between male and female population. Thus we denote by $v_i(t), w_i(t), y_i(t), z_i(t)$ the size of male population belonging respectively to P_v, P_w, P_y, P_z and of age in the age group indexed by i, at time t.

The corresponding female population is denoted by over bar letters: $\overline{v_i}(t)$, and so on.

TABLE 1

i: age group index	age	l_i: length of the age group i
1	from 0 to 5 years included	6
2	» 6 » 10 » »	6
3	» 11 » 13 » »	5
4	» 14 » 18 » »	5
5	» 19 » 24 » »	3
6	» 25 » 34 » »	10
7	» 35 » 54 » »	20
8	» 55 » 64 » »	10
9	» 65 and over	—

Time t belongs to a fixed interval in the set of integers and is assumed to vary from year to year.

The equations given below will simply state that, for instance, the number of males employed in agriculture of age group i (v_i) at time $t+1$ is the sum of the following terms:

i) those parts of $v_i(t), w(t), y_i(t)$ and $z_i(t)$ still living, still belonging to age group i working in agriculture at time $t+1$;

ii) those parts of $v_{i-1}(t), w_{i-1}(t), y_{i-1}(t)$ and $z_{i-1}(t)$ still living and belonging to age group i and working in agriculture at time $t+1$;

iii) the net migrated population from/in P during year t, working in agriculture and belonging to age group i.

In order to represent i) and ii), we have to look for the birthrate and the surviving-rate coefficients; so we take $b^i_v(t)$ and $d^i_v(t)$ as such coefficients for male population employed in agriculture, belonging to age group i at time t. Furthermore, in order to represent ii) we need to know the amount of population that will pass from age group i to age group $i+1$ in year t; this is done by forward linear interpolation between the two age groups, because the crude assumption that age distribution is constant in each age group would be quite imprecise. Therefore, the population (dead or alive) going into age group $i+1$ at time $t+1$ say, in agriculture, will be:

$$\frac{v_i(t)}{l_i} + \frac{(l_i-2)\,(l_i\,v_{i+1}(t) - l_{i+1}\,v_i(t))}{l_i\,l_{i+1}\,(l_i+l_{i+1})}$$

where l_i is the length of age group i.

Finally, in order to represent i), ii) and iii), we shall need the following symbols:

— $m^i_{ab}(t)$, $a, b, = v, w, y, z$, to denote the rate of internal migration of the population from part b to part a, belonging to age group i at time t;

— $n^i_a(t)$, $a = v, w, y, z$, to denote the number of net migrated population from/in part a, belonging to age group i at time t.

Thus, starting with male population, we can write the following equations:

$$h_i(t+1) = \sum_{\substack{a=v,w\\y,z}} m^i_{ha}(t)\, d_a^{i-1}(t)\left(\frac{a_{i-1}(t)}{l_{i-1}} + \frac{(l_{i-1}-2)\,(l_{i-1}\,a_i(t) - l_i\,a_{i-1}(t))}{l_{i-1}\,l_i\,(l_{i-1}+l_i)}\right) +$$

$$+ \sum_{\substack{a=v,w\\y,z}} m^i_{ha}(t)\, d^i_a(t)\left(a_i(t) - \frac{a_i(t)}{l_i} - \frac{(l_i-2)\,(l_i\,a_{i+1}(t) - l_{i+1}\,a_i(t))}{l_i\,l_{i+1}\,(l_i+l_{i+1})}\right) +$$

$$+ n^i_h(t) \tag{1}$$

where $h = v, y, z$, $i = 4, 5, 6, 7, 8$ when $h = v, y$ and $i = 2, \ldots, 8$ when $h = z$; furthermore:

$$z_1(t+1) = d^i_z(t)\left(z_1(t) - \frac{z_1(t)}{l_1} - \frac{(l_1-2)\,(l_1\,z_2(t) - l_2\,z_1(t))}{l_1\,l_2\,(l_1+l_2)}\right) +$$

$$+ \sum_{i=4}^{7} \sum_{\substack{a=v,w\\y\,z}} b^i_a(t)\, a_i(t) + n^1_z(t) \tag{2}$$

$$z_9(t+1) = \sum_{\substack{a=v,w\\y,z}} d^8_a(t)\left(\frac{a_7(t)}{l_7} + \frac{(l_7+2l_8-2)\,(l_7\,a_8(t) - l_8\,a_7(t))}{l_7\,l_8\,(l_7+l_8)}\right) +$$

$$+ d^9_z(t)\, z_9(t) + n^9_z(t). \tag{3}$$

We note that (1) represents 17 equations, (2) the births and (3) the going out of work because of age limits. In (1) we assume that there is no work for age group 1, 2, 3 and 9, and in (3) we use backward linear interpolation. By putting a bar over the variables, (1), (2) and (3) represent the female population, too. So, we have 38 equations with 46 variables. The missing equations refer to student population and will be considered below.

Usually, the available data on population migration are not divided into age and work groups: hence we assume that only a global number $m(t)$, and the total net migration at year t, are known and we replace each $n^i_a(t)$ in (1), (2) and (3) with a fixed fraction of $m(t)$:

$$n^i_a(t) = \beta^i_a \, m(t) \tag{4}$$

where the β^i_a's are suitable constants.

Equations (1), (2), (3), (4) can be written in compact form as follows:

$$\begin{bmatrix} v(t+1) \\ y(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} A_{vv}(t) & A_{vy}(t) & A_{vz}(t) \\ A_{yv}(t) & A_{yy}(t) & A_{yz}(t) \\ A_{zv}(t) & A_{zy}(t) & A_{zz}(t) \end{bmatrix} \begin{bmatrix} v(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_{vw}(t) & B_{vm} \\ B_{yw}(t) & B_{ym} \\ B_{zw}(t) & B_{zm} \end{bmatrix} \begin{bmatrix} w(t) \\ \\ m(t) \end{bmatrix} \tag{5}$$

where the blocks in the coefficient matrices can be easily deduced from (1) to (4).

The model looks like a linear time-varying system, but, as we shall see in Section 4, some coefficients depend on other variables in the system, thus destroying linearity.

Now, we are going to describe the school subsystem.

For sake of simplicity, we are not going to distinguish between public and private schools; this seems to be justified here, because in Italy private schools represent only 5% of the total amount of schools and they are quite similar to public school. However, it will not be difficult to take this into consideration.

We denote by $s_i(t)$ and $\overline{s_i}(t)$ the number of male and female students at school level i in year t, where level i is chosen according to Table 2 below.

TABLE 2

Level i	school year
1	first year of elementary shool
.	. . . . .
5	fifth » » » »
6	first year of secondary school
.	. . . . .
8	third » » » »
9	first year of high school
.	. . . . .
13	fifth » » » »
14	first year of university
.	. . . .
19	sixth » » »

We assume that population can leave school only when ending a school cycle, that is only when $i=5, 8, 13$; then we add the probability of leaving university at the first year, that is when $i=14$, because from our data this turned out to be very important. For other values of i, one can leave school only by death.

The rate of admittance from level i to level $i+1$ is $p_i(t)$ for male students and $\bar{p}_i(t)$ for female students. Then, we assume that a student who is not admitted to the next form will stay one more year in level i and that this happens only once for each student.

For the levels where leaving school is not supposed to take place, the number of students will be given by:

$$s_i(t+1)=p_{i-1}(t)\, d^k{}_w(t)\, s_{i-1}(t)+(1-p_i(t))\, d^k{}_w(t)\, s_i(t) \qquad (6)$$

$$i \neq 1, 5, 8, 13, 14$$

where k is the index of the age group corresponding to group level i; thus $k=2$ for $i=1, \dots, 5$, $k=3$ for $i=6, 7, 8$, and so on.

For the remaining levels we have:

$$s_i(t+1)=(1-\sum_{a=y,v,z} m^{i-1}_{a,s}(t))\, p_{i-1}(t)\, d^k{}_w(t)\, s_{i-1}(t)+$$

$$+(1-p_i(t))\, d^k{}_w(t)\, s_i(t) \qquad (7)$$

with $i=5, 8, 13, 14$ and correspondingly $k=2, 3, 4, 5$.

$$s_1(t+1)=\alpha_1(t)\, d^1_z(t)\left(\frac{z_1(t)}{l_1}+\frac{(l_1-2)\,(l_1 z_2(t)-l_2 z_1(t))}{l_1 l_2 (l_1+l_2)}\right)+$$

$$+(1-p_1(t))\, d^1_w(t)\, s_1(t) \tag{8}$$

where $\alpha_i(t)$ represents the fraction of population not going to school at the age of six, in spite of law. Together with similar equations for female students, (6), (7), (8) are 38 equations representing a linear time-varying system; however, as in the previous case, we shall see that coefficients depend on the state, and this destroys linearity.

Model (6), (7), (8) is linked to model (5) through equation (8); another link arises when obtaining the population that goes from school into other parts of the model.

Bearing in mind the choice of the age groups (see Table 1) and the assumption that leaving school takes place only at the end of a school cycle, the comparison between (1) and (7) gives:

$$m^i_{hw}(t)\, d^i_w(t)\left(w_i(t)-\frac{w_i(t)}{l_i}-\frac{(l_i-2)\,(l_i w_{i+1}(t)-l_{i+1}\; w_i(t))}{l_i l_{i+1}(l_i+l_{i+1})}\right)+$$

$$+m^i_{hw}(t)\, d_w^{i-1}(t)\left(\frac{w_{i-1}(t)}{l_{i-1}}+\frac{(l_{i-1}-2)\,(l_{i-1} w_i(t)-l_i w_{i-1}(t))}{l_i l_{i-1}(l_i+l_{i-1})}\right)=$$

$$m^k_{h,s}(t)\, d_w^{i-1}(t)\, s_k(t) \tag{9}$$

for $h=v, y, z,$ $i=3, 4$, and, correspondingly, $k=5, 8$.

Taking into consideration the probability of leaving university after the first year, we also get:

$$m^5_{hw}(t)\, d^5_w(t)\,(w_5(t)-\frac{w_4(t)}{l_4}-\frac{(l_4+2l_5-2)\,(l_4 w_5(t)-l_5 w_4(t)}{l_4 l_5 (l_4+l_5)}\Bigg)$$

$$+m^5_{hw}(t)\, d^4_w(t)\left(\frac{w_4(t)}{l_4}+\frac{(l_4-2)\,(l_4 w_5(t)-l_5 w_4(t))}{l_4 l_5 (l_4+l_5)}\right)=$$

$$=m^{13}_{hs}(t)\, d^4_w(t)\, s_{13}(t)+m^{14}_{hs}(t)^5{}_w(t)\, s_{14}(t). \tag{10}$$

Hence the terms $B_{aw}(t)\, w(t)$ in (5) can be replaced by the corresponding terms in the r. h. s. of (9), (10), thus obtaining:

$$\begin{bmatrix} v(t+1) \\ y(t+1) \\ z(t+1) \end{bmatrix} =$$

$$= \begin{bmatrix} A_{vv}(t) & A_{vy}(t) & A_{vz}(t) \\ A_{yv}(t) & A_{yy}(t) & A_{yz}(t) \\ A_{zv}(t) & A_{zy}(t) & A_{zz}(t) \end{bmatrix} \begin{bmatrix} v(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_{vs}(t) & B_{vm} \\ B_{ys}(t) & B_{ym} \\ B_{zs}(t) & B_{zm} \end{bmatrix} \begin{bmatrix} s(t) \\ m(t) \end{bmatrix} \tag{11}$$

Together with (11) we also rewrite (6), (7), (8) in compact form:

$$s(t+1) = A_{ss}(t)\, s(t) + B_{sz}(t)\, z(t) \tag{12}$$

where matrices A_{ss} and B_{sz} are easily found from (6), (7) and (8).

Thus (11) and (12) constitute the model of the whole population; the structure of the model is the one already anticipated and depicted in Section 2, and consists of 76 equations.

Furthermore, we note that in (11) and (12) the age groups in school, w_i, have been eliminated. If one is interested in, it is possible to get them through the assumption that nobody leaves school at the non allowed levels and that nobody stays in the same level more than two years.

Thus it is easy to verify that:

$$w_2(t) = \sum_{i=1}^{5} s_i(t) -$$

$$- \sum_{i=1}^{5} (1 - p_i(t+i-6)) \prod_{k=1}^{5} d^2_w(t+k-6)\, s_i(t+i-6) \tag{13}$$

$$w_3(t) = \sum_{i=6}^{8} s_i(t) +$$

$$+ \sum_{i=1}^{5} (1 - p_i(t+i-6)) \prod_{k=i}^{5} d^3_w(t+k-6)\, s_i(t+i-6) +$$

$$+ \sum_{i=6}^{8} (1 - p_i(t+i-9)) \prod_{k=i}^{8} d^k_w(t+k-8)\, s_i(t+i-8) \tag{14}$$

$$w_4(t) = \sum_{i=9}^{13} s_i(t) + \sum_{i=6}^{8} (1-p_i(t+i-9)) \prod_{k=i}^{8} d^4_w(t+k-9)\, s_i(t+i-9) +$$

$$(1-p_i(t+i-14)) \prod_{k=i}^{13} d^k_w(t+k-14)\, s_i(t+i-14) \qquad (15)$$

$$w_5(t) = \sum_{i=14}^{19} s_i(t) +$$

$$+ \sum_{i=8}^{18} (1-p_i(t+i-14)) \prod_{k=i}^{8} d^5_w(t+k-14)\, s_i(t+i-14) \qquad (16)$$

Hence, if w_i must formally appear in the model, it is necessary to add 19 further state variables to represent the delays appearing in (13) to (16).

4. The model.

Relationship (11) and (12) we got in the previous section hide a lot of complex social and economical phenomena in their coefficients. In this section we are going to analyze these coefficients and try to relate them with some control variables.

i) Starting with block matrix A_{yv}, we assume that direct migration from farmwork to other kinds of work is mainly linked to $u_1(\cdot)$, i. e. the industrial investment. The dependence is clearly dynamic with finite memory, because investments generate work possibilities directly and by various kinds of induction, the latter with more or less delay. Furthermore, we assume that over the time interval involved (about 30 years in our application of Section 5) no structural changes will occur in the society; thus the model is time invariant. At least, we invoke the choice of simplicity to linearize the aforesaid dependence. Thus we get:

$$m^i_{yv}(t) = \frac{l}{v_i(t)} \left(\overline{m}^i_{yv} + \sum_{k=t-l}^{t} a^i_k\, u_1(t-k)\right) \quad i=3,\ldots,7 \qquad (17)$$

where $\overline{m}^i_{yv}$ is a constant arising by linearization, a_k are suitable coefficients and l is the length of the interval on which an investment affects the occupation (length of memory).

Of course, we know that other variables such as wages, prices, housing policy and so on play a role in the phenomenon, [10]; however our assumption is simple, significant and, above all, in full agreement with our data.

Let us rewrite (17) in compact form as follows:

$$A_{yv}\, v\,(t) = \bar{A}_{yv} + \sum_{k=t-1}^{2} A_k\, u_1\,(t-k) \tag{18}$$

ii) Looking at block A_{zv}, we note that underemployment in agriculture can be explained by the following fact: when farmers leave their farms to go to work elsewhere, the land often remains uncultivated, thus reducing the use of labourers. This seems a mainly static dependence; thus we put

$$A_{zv}\,(t) = k\, A_{yv}\,(t) \tag{19}$$

where k is a suitable constant. From relation

$$m^i_{vv} + m^i_{yv} + m^i_{zv} = 1 \tag{20}$$

block A_{vv} is found, too.

iii) Looking at the second block-column of A in (11), we assume that the number of workers leaving their work to go to work in agriculture is negligeable; thus

$$A_{vy}\,(t) = 0 \tag{21}$$

The passage from non-agricultural employment to unemployement depends both on the investments and political decisions of the Government and Trade Unions. The latter presence acts to keep coefficients m^i_{zy} at zero; then, we assume that this result is obtained, as is the case in Italy, and put

$$A_{zy}\,(t) = 0 \tag{22}$$

It follows that

$$m^i_{yy}\,(t) = i \tag{23}$$

thus $A_{yy}\,(t)$ is known.

iv) Going forward to the third block-column of A, and bearing in mind the permanent decrease of the employment in agriculture, we assume that coefficients $m^i_{vz}\,(t)$ are non-zero only when an intervention ad hoc by the Government takes place. Denoting with $u_2(t)$ the public investment for the employment in agriculture during yeart t, and talking as in i), we get

$$A_{vz}\, z(t) = \overline{A}_{vz} + \sum_{k=t-m}^{t} B_k\, u_2(t-k) \tag{24}$$

where $\overline{A}_{vz}$, B_k are constants and m has the meaning of length of a finite memory system.

Once again a reasoning like in i) brings us to put

$$A_{yz}\, z(t) = \overline{A}_{yz} + \sum_{k=t-n}^{t} C_k\, u_1(t-k) \tag{25}$$

with obvious meaning of symbols. Finally, from the obvious relationships

$$m^i_{vz}(t) + m^i_{yz}(t) + m^i_{zz}(t) = 1 \tag{26}$$

we get for A_{zz} the following expression:

$$A_{zz}\, z(t) = \overline{A}_{zz} - \sum_{k=t-n}^{t} \left(B_k\, u_2(t-k) + C_k\, u_1(t-k)\right) \tag{27}$$

where we assume $n > m$, as usually happens, and $b_{m+1}, \ldots, b_n = 0$.

v) The data on the rates of admittance from one form to the next one, in the Italian school, show that any coefficients $p_i(t)$ appearing in matrix A_{ss} can be assumed to be linear functions of a single variable, i. e. of $u_3(t)$, which represents a measure of difficulty in the school. We generalize this by putting

$$p_i(t) = \overline{p}_i + q_i\, u_3(t) \tag{28}$$

in our model, taking variable $u_3(t)$ as one of the inputs.

vi) To complete our investigation on A_{ss}, we have to consider also the coefficients of the passage from one school cycle to the other, that is

$$m_{12}(t) \triangleq 1 - m^5_{vs}(t) - m^5_{ys}(t) - m^5_{zs}(t) \tag{29}$$

$$m_{23}(t) \triangleq 1 - m^8_{vs}(t) - m^8_{ys}(t) - m^8_{zs}(t) \tag{30}$$

$$m_{34}(t) \triangleq 1 - m^{13}_{vs}(t) - m^{13}_{ys}(t) - m^{13}_{zs}(t) \tag{31}$$

respectively for the passage from Elementary to Secondary School, from Secondary to High School and from High School to University.

As regards m_{12}, we assume that almost only peasants can try to avoid the secondary school for their sons, in spite of law; hence, with the usual hypotheses of time-invariance, finiteness of memory and linearization, we put:

$$m_{12}(t)=\bar{m}_{12}-\sum_{k=t-q}^{t} f_k \frac{\Sigma\, v_i(t-k)}{\Sigma\,(v_i(t-k)+y_i(t-k))} \tag{32}$$

Being, in virtue of law, $m^5_{ys}=m^5_{vs}=0$, also m^5_{zs} is obtained from (29):

$$m^5_{zs}(t)=\bar{m}^5_{zs}-\sum_{k=t-q}^{t} f_k \frac{\Sigma\, v_i(t-k)}{\Sigma\, v_i(t-k)+\Sigma\, y_i(t-k)} \tag{33}$$

where:

$$\bar{m}^5_{zs}=1-\bar{m}_{12} \tag{34}$$

As regards the coefficients in the l. h. s. of (30) and (31), we do not attempt an interpretation, which would be a big matter of discussion, but simply note that, in Italy, m_{23} and m_{34} turn out to depend strongly on the relative number of workers; we generalize the Italian situation by putting:

$$m_{23}(t)=\bar{m}_{23}+\sum_{k=t-r}^{t} h_k \frac{\Sigma\,(v_i(t-k)+y_i(t-k))}{\Sigma\,(v_i(t-k)+y_i(t-k)+z_i(t-k))} \tag{35}$$

$$m_{34}(t)=\bar{m}_{34}+\sum_{j=t-s}^{t} k_j \frac{\Sigma\,(v_i(t-j)+y_i(t-j))}{\Sigma\,(v_i(t-j)+y_i(t-j)+z_i(t-j))} \tag{36}$$

We further assume, as in previous cases, that m^8_{ys}, m^{13}_{ys} depend on the industrial investments in the usual manner:

$$m^8_{ys}(t)=\bar{m}^8_{ys}+\sum_{k=t-n}^{t} l_k\, u_1(t-k) \tag{37}$$

$$m^{13}_{ys}(t)=\bar{m}^{13}_{ys}+\sum_{k=t-\alpha}^{t} n_k\, u_1(t-k) \tag{38}$$

Thus, by putting $m^8_{vs}=m^{13}_{vs}=0$, all the coefficients in (30) and (31) are linked to other variables of the model:

$$m^{8}_{zs}(t)=\bar{m}^{8}_{zs}-\sum_{k=t-b}^{t}\left(h_k \frac{\Sigma\,(v_i(t-k)+y_i(t-k))}{\Sigma\,(v_i(t-k)+y_i(t-k)+z_i(t-k))}+\right.$$
$$\left.+l_k\,u_1(t-k)\right) \tag{39}$$

$$m^{13}_{zs}(t)=\bar{m}^{13}_{zs}-\sum_{j=t-c}^{t}\left(k_j \frac{\Sigma\,(v_i(t-j)+y_i(t-j))}{\Sigma\,(v_i(t-j)+y_i(t-j)+z_i(t-j))}+\right.$$
$$\left.+n_j\,u_1(t-j)\right) \tag{40}$$

with the same use of symbols as before.

Finally, we need to investigate the coefficients relative to the end of the university cycle; again applying previous reasoning, we put

$$m^{19}_{ys}(t)=\bar{m}^{19}_{ys}+\sum_{k=t-d}^{t} q_k\,u_1(t-k) \tag{41}$$

$$m^{19}_{vs}(t)=0$$

$$m^{19}_{zs}(t)=1-m^{19}_{ys}(t)=m^{19}_{zs}-\sum_{k=t-d}^{t} q_k\,u_1(t-k) \tag{42}$$

vii) The last step is to replace (18) to (42) in (11) and (12) and add the necessary number of further state variables to eliminate the delays appearing in the r. h. sides of (18), (24), (25), (27), (32) to (42). If d denotes the vector of these new state variables, and $x=$ col (v, y, z, d), then the resulting equations can be written in a compact form as follows:

$$x(t+1)=f(x(t))+\sum_{i=1}^{3} g_i(x(t))\,u_i(t)+bm(t) \tag{43}$$

Functions f and g are nonlinear in x, because products and quotients among components of x appear; conversely, the model is linear in inputs u_i as an obvious consequence of previous linearizations. Net migration m enters linearly in the model and it has been hold distinct from other inputs because it plays the role of a disturbance, not as a control variable.

Equations (43) looks like the discrete-time counterpart of a continuous-time « linear analytic » system [3], whose control problems were recently investigated with interesting results [4]; so, it seems natural to look for the extension of that results to discrete-time systems, like the present model.

Before ending this section, we note that in (43) the non linearity in the state could be locally eliminated by linearization; thus the model would become a bilinear system.

The possibilities of replacing a linear analytic system with a bilinear one have been widely investigated in the continuous-time case [5, 6]; however we note two facts:

i) a look at the Italian available data shows that the inputs are subject to lower variations with time than some state variables;

ii) it is possible and, we think, reasonable to assume that inputs enter linearly in the model, while the analysis of this section shows that this is not the case for the state.

5. A first application to Italy.

Now, we are going to present a first application of the model to the Italian society. After a brief note on the problems arising in collecting the required data and a view of the data we have been able to collect, (shown in fig.'s 4, 5, 6), we shall use these data to obtain a first validation of the model, aggregating some equations, and, at the end, we shall make a few remarks on the results thus obtained.

The data on the size of population divided up according to sex and age groups present no problems, so we are not going to spend any more words on this subject. Starting with (17), we shall use official data [7, 8] to get the values of the industrial investments (u_1), number of workers in agriculture and in other fields, and underemployment in agriculture. These data will not be divided into age groups, thus for the moment we shall be forced to the aggregation with respect to index i in (17).

Therefore, by a stepwise regression, we determinate for m_{yv} the following expression:

$$m_{yv}(t) = 2.96\, u_1(t) - 0.17\, u_1(t-1) - 0.16\, u_1(t-2) - \\ - 0.15\, u_1(t-3) - 2.49\, u_1(t-4) \qquad (44)$$

where u_1 is in millions of U. S. dollars 1963. Fisher's test shows that the regression is significant at the 1% level. Yet by the same data we

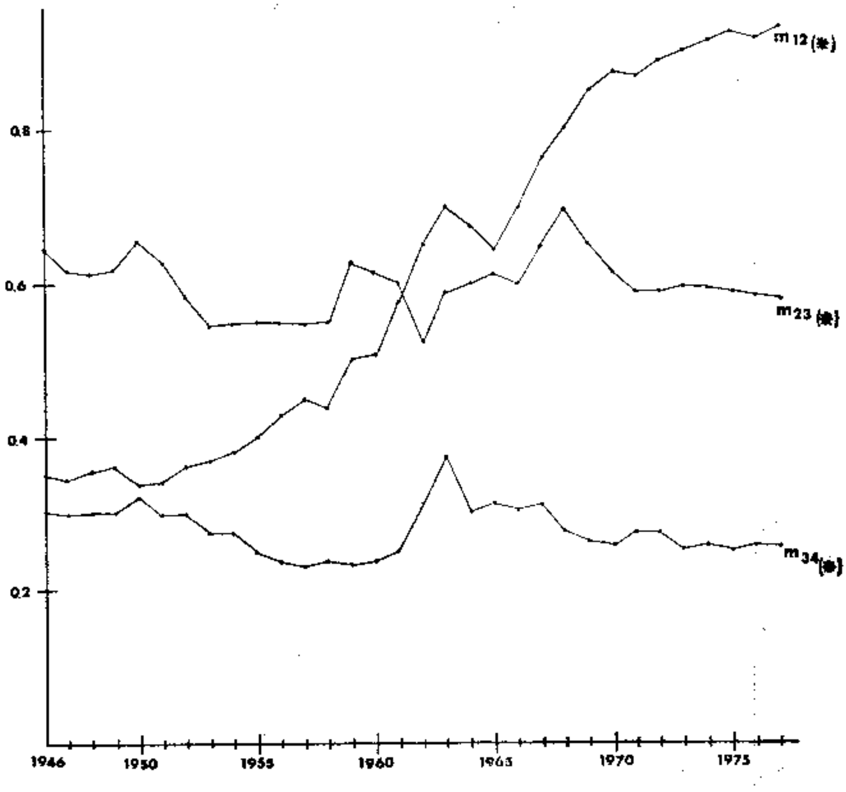

fig. 4

(*) taking into account the first-year abandons.

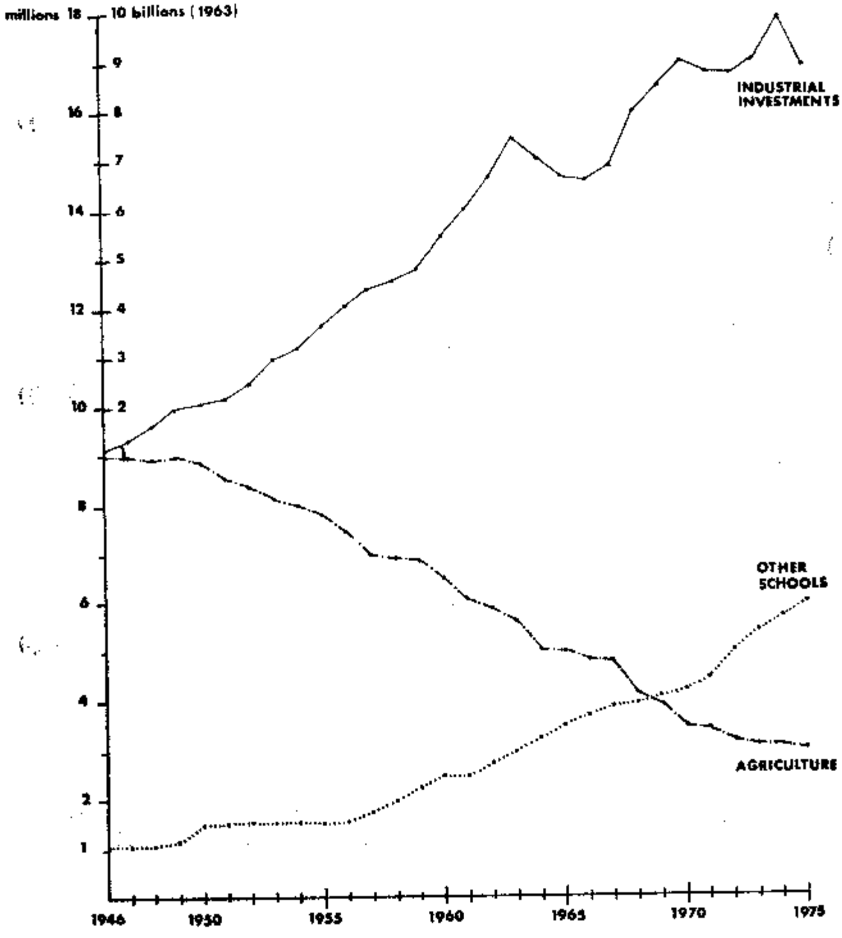
millions
10 billions (1963)
INDUSTRIAL INVESTMENTS
OTHER SCHOOLS
AGRICULTURE
1946
1950
1955
1960
1965
1970
1975

fig. 5

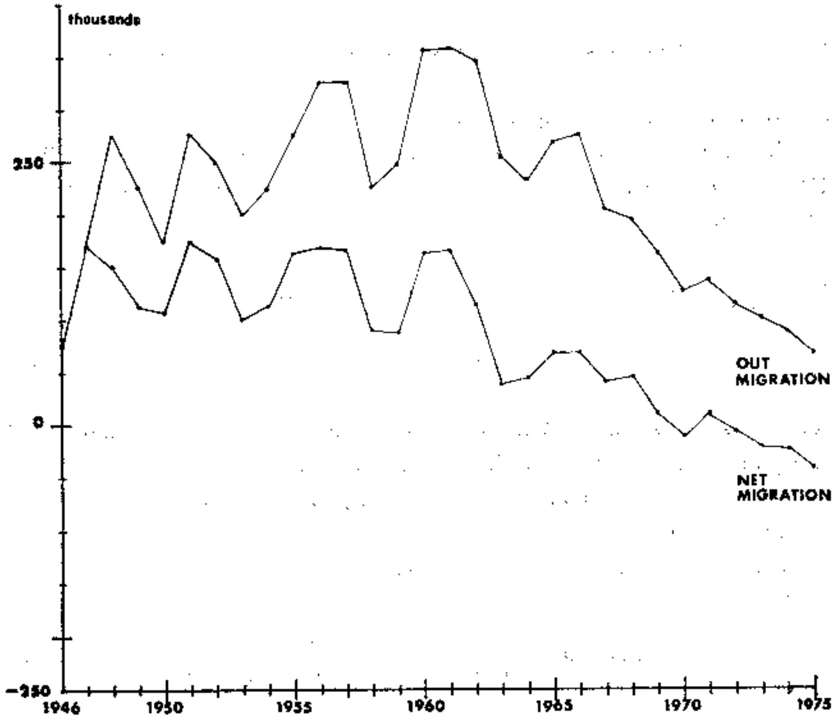
thousands
250
0
-250
1946
1950
1955
1960
1965
1970
1975
OUT
MIGRATION
NET
MIGRATION

fig. 6

see that k is less than 10^{-2} in (19), thus we put $A_{zv}=0$, $m_{vv}=1-m_{yv}$.

We proceed in a similar manner with respect to (24), (25) and (27) obtaining

$$m_{vz}\, z\,(t)=0.027+0.114\, u_2\,(t)+0.014\, u_2\,(t-1) \tag{45}$$

$$m_{yz}\, z\,(t)=0.102+0.097\, u_1\,(t)+0.063\, u_1\,(t-1)+0.019\, u_1\,(t-2) \tag{46}$$

yet with a 1% level of significance.

The coefficients in (28) have been easily determinated, together with $u_3\,(t)$ from data in [9]; the same data allow the determination of m_{12} in (32) as follows:

$$m_{12}=(1320-190\, v\,(t)+57\, v\,(t-1)+171\, v\,(t-2)-$$
$$-207\, v\,(t-3)+5\, v\,(t-4))\; 10^{-3} \tag{47}$$

where v is in millions. The level of significance is better than 1%; looking at the importance of this coefficient, we show the measured and the evaluated values in fig. 7. As a consequence thereof, m_{zs} is known from (33), (34).

In the same manner we obtain, for (35), (36):

$$m_{23}=0.621-0.127\,\frac{v\,(t-1)+y\,(t-1)}{v\,(t-1)+y\,(t-1)+z\,(t-1)} \tag{48}$$

$$m_{34}=0.304-0.021\,\frac{v\,(t-1)+y\,(t-1)}{v\,(t-1)+y\,(t-1)+z\,(t-1)}-$$
$$-0.017\,\frac{v\,(t-2)+y\,(t-2)}{v\,(t-2)+y\,(t-2)+z\,(t-2)}. \tag{49}$$

The measured and the evaluated data are compared in fig. 8 and in fig. 9, yet showing a good agreement.

Going forward to (37) and (38), at present we have no information on the employment of students after their degree; hence we make a crude semplification assuming:

$$m^{8}_{zs}=m^{13}_{zs}=m^{19}_{zs}=0 \text{ and thus obtaining}$$

$$m^{8}_{ys}=1-m_{23}$$

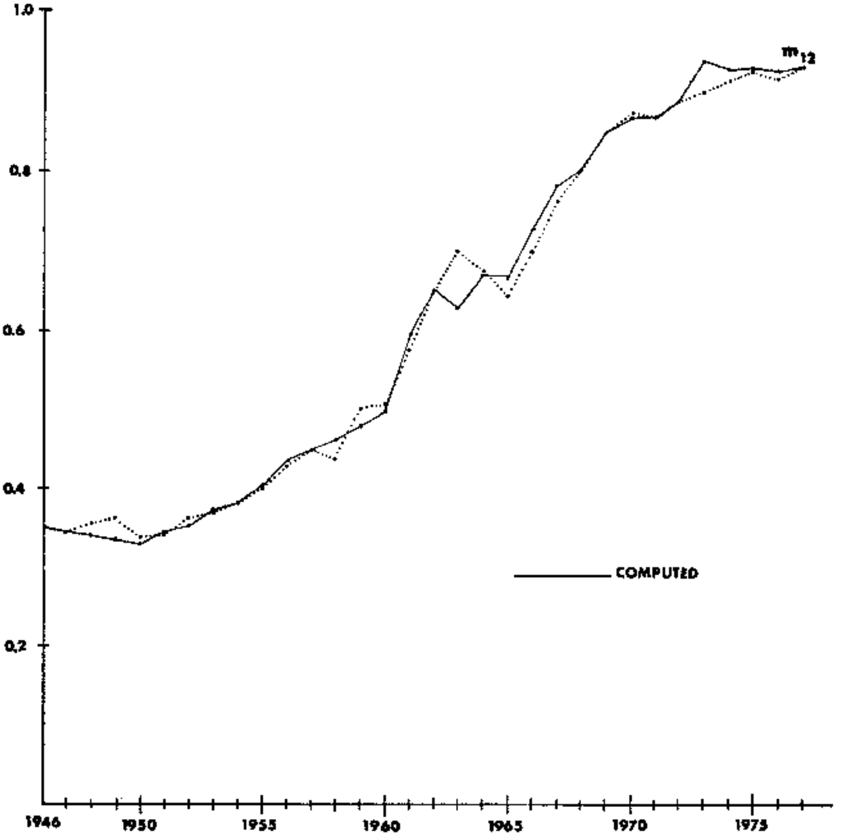

fig. 7

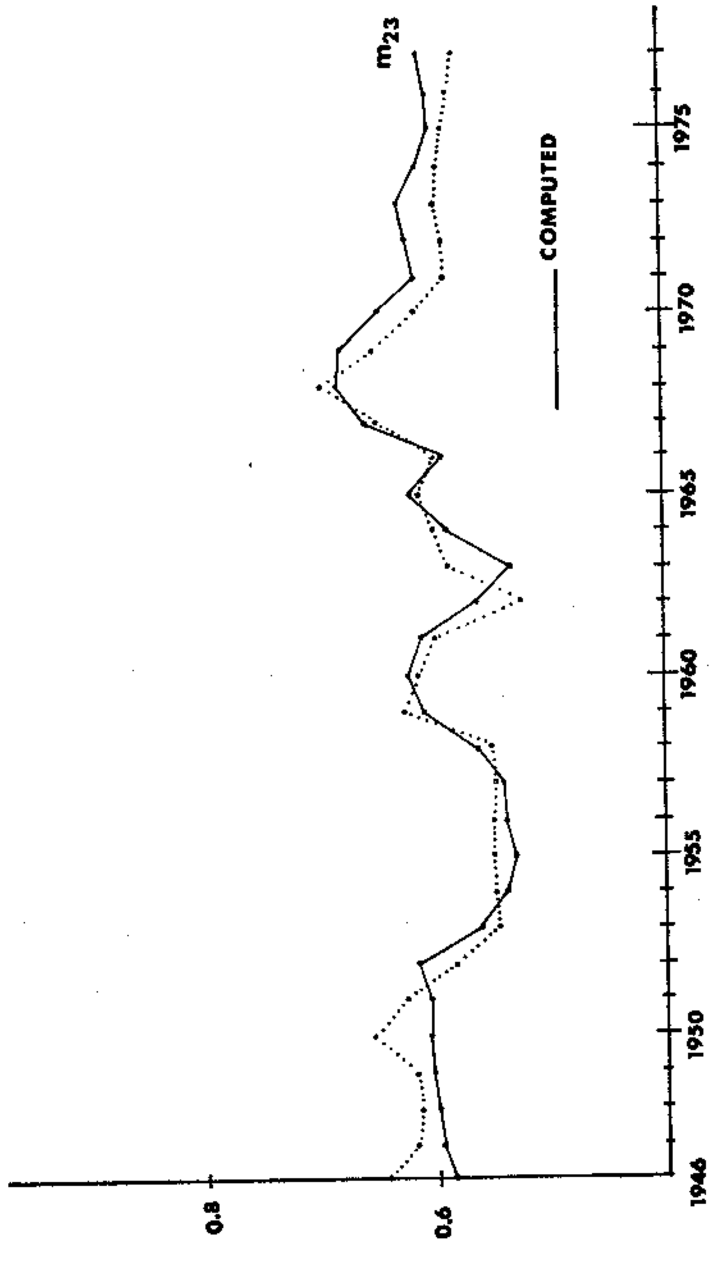

fig. 8

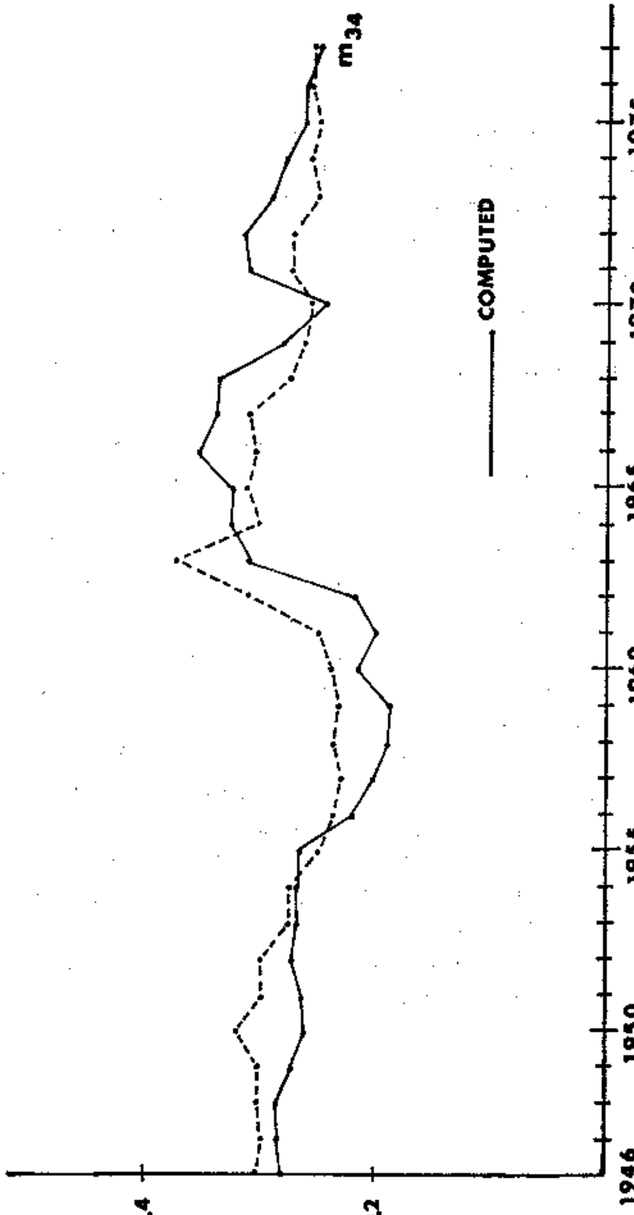

fig. 9

$$m^{13}_{ys} = 1 - m_{34}$$

$$m^{19}_{ys} = 1.$$

Thus, the aggregated and simplified form of the model has been completed with data. In spite of the many simplifications, the resulting test is in agreement with the data: after testing the model over a period of twenty years, i. e. from 1956 to 1975, we obtained the values of the aggregated state variables in fig. 10 and in fig. 11. Although this is only a first test, the result is a good validation for the model.

And now a few remarks on the resulting description of the phenomenon.

After World War One, in Western Europe the abandon of farms showed a significant increase. This did not happen in Italy because of the fascist government, that acted to keep the number of workers in agriculture at a high constant level (over nine millions, i. e. half of the whole labour force).

After World War Two, Italy underwent a settling period of about five years; then, in 1949-50, a fast growing of the industrial economy started. At the same time the abandon of forms started and - as we have shown - the link between these two phenomena is really strong. With one or two years' delay the increase in m_{12}, i. e. the rate of passage from Elementary to Secondary School, started, and - as we have seen - this fact is strongly dependent on the relative number of workers in agriculture. The other coefficients, i. e. m_{23} and m_{34}, are almost constant; hence, the increase in the number of students is mainly due to m_{12}, that is to the abandon of agriculture, then to the industrial development.

The analysis of social problems arising from these facts is out of the purpose of this paper; we shall only note that the sons of the internal migration we depicted before arrived to be at the end of their university studies in 1968.

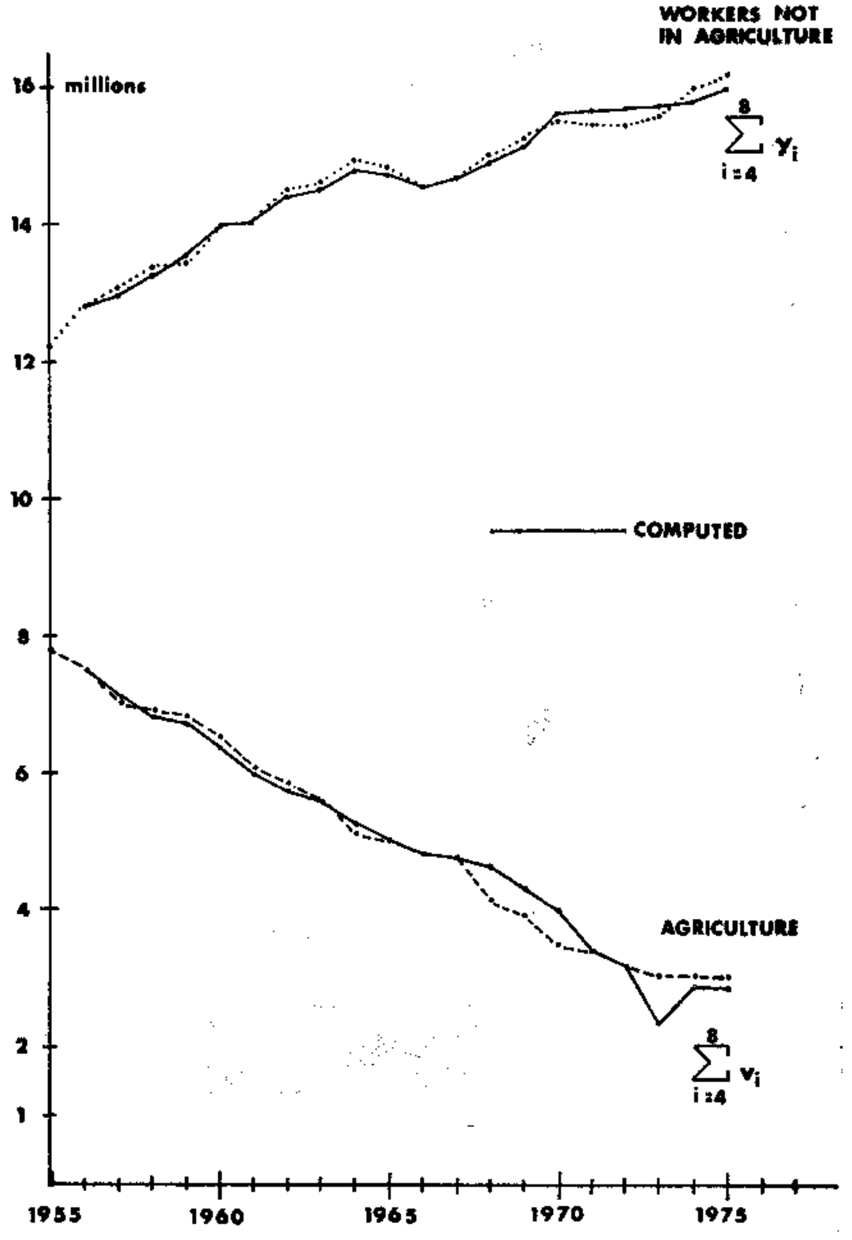
WORKERS NOT
IN AGRICULTURE
millions
16
14
12
10
8
6
4
2
1
$\sum_{i=4}^{8} y_i$
COMPUTED
AGRICULTURE
$\sum_{i=4}^{8} v_i$
1955
1960
1965
1970
1975

fig. 10

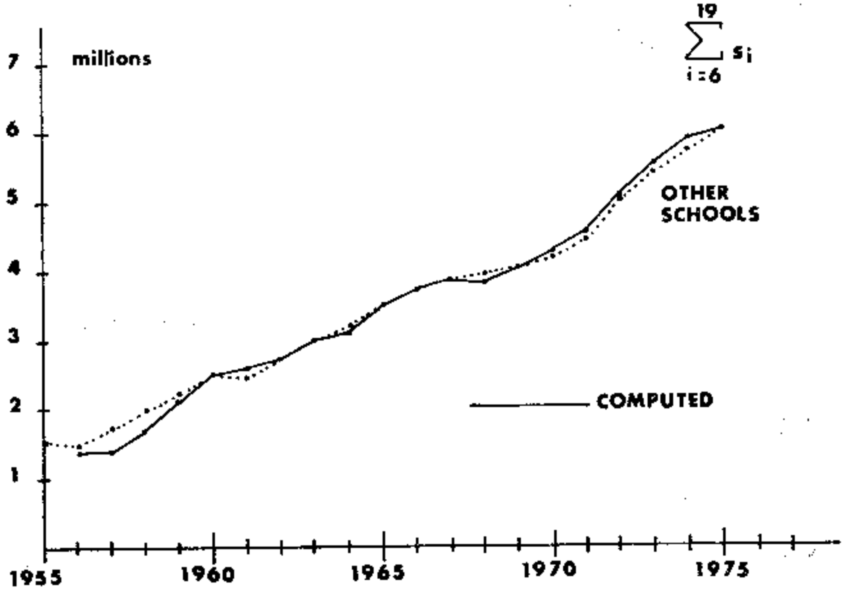
millions
7
6
5
4
3
2
1
$\sum_{i=6}^{19} s_i$
OTHER SCHOOLS
COMPUTED
1955
1960
1965
1970
1975

fig. 11

REFERENCES

[1] P. H. LESLIE: *On the use of matrices in certain population mathematics*, Biometrika, v. 33, pp. 183-212, 1945.

[2] N. KEYFITS: *Introduction to the mathematics of population*, Addison-Wensley, U. S. A., 1968.

[3] R. W. BROCKETT: *Volterra series and geometric control theory*, Automatica, v. 12, pp. 167-176, 1976.

[4] A. ISIDORI, A. J. KRENER, C. GORI-GIORGI, S. MONACO: *Nonlinear decoupling via feedback: a differential geometric approach*, submitted to IEEE Trans. on Automatic Control.

[5] A. J. KRENER: *Bilinear and nonlinear realizations of input-output maps*, IEEE Trans. on A. C., 1977.

[6] C. GORI-GIORGI, A. ISIDORI, A. RUBERTI: *A survey of some recent applications of the theory of bilinear systems to the analysis of nonlinear systems*, 3rd Italian-Polish conference on applications of system theory to economy, management and technology, Warszawa 1977.

[7] ISTAT: Annuario di statistiche del lavoro, vol. I-XVIII.

[8] ISTAT: Annuario di statistiche industriali, vol. I-XXI.

[9] ISTAT: Annuario statistico dell'istruzione, vol. I-XXIX.

[10] *Multi-level planning: case studies in Mexico*, L. M. GOREUX and A. S. MANNE editors, North-Holland, U. S. A., 1973.