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UNIFORMITY AND REVERSIBILITY CONDITIONS FOR LINEAR TIME - VARYING SYSTEMS

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Abstract. In Section 3 conditions are given on a linear system to be uniform, i. e. such that every input-output pair starting at time t is the restriction of another pair starting at any time $t_0 < t$. In Section 4 conditions are given on a linear uniform system to be reversible, i. e. to admit a state-representation with a nonsingular transition matrix. In Section 5 an example is presented.

1. Introduction.

The concept of system as a set of input-output pairs was investigated by several authors, giving rise to definitions which, with slight differences, can be divided into two groups: definitions assuming a system to be a set of input-output pairs defined on the whole time set [4, 5], and definitions describing a system as the collection of all the sets of input-output pairs starting at each time [1, 2, 3]. The first definition is justified by practice: most objects are described by differential equations defining the input-output pairs on the whole time set. However it is quite easy to find physical objects to be considered « systems » which are not systems in the first sense (see [3] and the example in Section 5); hence the second kind of definition will be assumed in this paper.

When such choice is made, the problem naturally arises of finding conditions under which a system can be described by a unique set of input-output pairs defined on the whole time set. This problem is solved

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by the « uniformity condition » given in Section 3 for the special class of linear time-varying systems. However the uniformity condition does not guarantee the existence of a differential equation; this problem is studied in Section 4, again for the linear time-varying systems: the key property is the « reversibility » already introduced in [6].

The theory is presented in the matrix language, which is more suitable for algorithmic implementations; however a formulation in the abstract algebraic language would be straightforward. Only the continuous-time case is considered, but the results hold unmodified for discrete-time systems (except the existence of the differential equation, of course).

2. Definitions and notations.

A *time set* T is given which is a $(\geq)$ simply-ordered subset of real numbers $\mathbf{R}$; for a $t_0 \in T$, $T(t_0)$ denotes the subset of T defined by

$$T(t_0) = \{t \mid (t \in T) \ \& \ (t \geq t_0)\} \tag{1}$$

U, Y respectively denote the nonempty sets of *input values* and *output values*; the *set of input-output pairs at time* t_0 is a relation

$$S(t_0) \subseteq U^{T(t_0)} \times Y^{T(t_0)} \tag{2}$$

which is thought as the collection of all the experiments on the system starting at t_0.

A *System* is a set

$$\begin{aligned} S = \{S(t_0), t_0 \in T \mid t_0, t_1 \in T, (u, y) \in S(t_0) \Rightarrow \\ \Rightarrow (u\,|_{T(t_1)}, Y\,|_{T(t_1)}) \in S(t_1)\} \end{aligned} \tag{3}$$

where $u\,|_{T(t_1)}$ denotes the restriction of the function u on the subset $T(t_1)$. Hence, by definition, a system has the quite obvious property that every restriction of one of its input-output pairs is again a pair of the same system. While this request seems a logical consequence of the orientation of time and therefore is assumed at the definition level, it is also possible to think systems having the symmetrical property that every pair comes out as the restriction of a preceeding pair. Such a property is very common in practice (e. g. systems represented by differential equations) but not all systems enjoy it [3]; hence it is assumed to characterize a class of systems called « uniform systems ». Formally a system S is *uniform* if

$$\begin{gathered}(u, y)\in S(t_1) \Rightarrow \exists\,(u', y')\in S(t_0): \\ (u'\,|_{T(t_1)}, y'\,|_{T(t_1)}) = (u, y), \quad \forall t_1\in T, \ \forall t_0<t_1.\end{gathered} \tag{4}$$

A system is *linear* if there exists a field k such that, $\forall t_0\in T$, $S(t_0)$ is a vector space over k.

A linear, nonanticipative, continuous-time, finite dimensional, proper system can be assigned by a state-representation of the kind:

$$x(t) = \Phi(t, t_0)\, x_0 + \int_{t_0}^{t} H(t, \tau)\, u(\tau)\, d\tau \tag{5}$$

$$y(t) = \Psi(t, t_0)\, x_0 + \int_{t_0}^{t} W(t, \tau)\, u(\tau)\, d\tau \tag{6}$$

where $x(t)\in \mathbf{R}^n$, $u(t)\in U = \mathbf{R}^p$, $y(t)\in Y = \mathbf{R}^q$ and the matrices are of proper dimensions.

Alinear system is *reversible* [6] if admits a state-representation (5, 6) such that

$$\det \Phi(t, t_0) \neq 0, \ \forall t_0\in T, \ \forall t\in T(t_0). \tag{7}$$

A system represented by differential equations is reversible, but the converse is not necessarily true; a reversible system is always uniform while the converse in general is not true.

3. The uniformity condition.

In this section the problem is faced of finding conditions on (5, 6) to represent an *uniform system*. The main tools are some structural properties of the state space which will be now defined. A state x is *strongly reachable at time t* if $\forall t_0<t$, $\exists\, x_0\, u$, such that (5) holds.

The set of strongly-reachable states is found in the following Lemma:

LEMMA 1. *The set $\mathcal{R}(t)$ of strongly-reachable states at time t is given by*

$$\mathcal{R}(t) = R\,[\Phi(t, t_m)] + R\left[\int_{t_m}^{t} H(t,\tau)\,H^T(t,\tau)\,d\tau\right] \tag{8}$$

where $R[\cdot]$ *denotes the range space and* t_m *is a small enough instant of time.*

Proof. First of all, remark that

$$R\,[\Phi(t, t_1)] \supseteq R\,[\Phi(t, t_0)], \quad \forall t_0 \leq t_1 \tag{9}$$

because

$$R\,[\Phi(t, t_0)] = \mathrm{R}\,[\Phi(t, t_1)\,\Phi(t_1, t_0)] =$$

$$= \Phi(t, t_1)\,R\,[\Phi(t_1, t_0)]. \tag{10}$$

Hence $\dim R\,[\Phi(t, t_0)]$ is a non-decreasing function of t_0; it follows that there exists an instant of time t_m such that:

$$R\,[\Phi(t, t_0)] = R\,[\Phi(t, t_m)], \quad \forall t_0 \leq t_m \tag{11}$$

If $x \in R\,[\Phi(t, t_m)]$, then $x \in R\,[\Phi(t, t_0)]$, $\forall t_0 < t_m$, hence

$$\forall t_0 < t, \ \exists x_0 : \ x = \Phi(t, t_0)\,x_0 \tag{12}$$

so that x is strongly reachable with $u = 0$. Moreover

$$x \in R\left[\int_{t_m}^{t} H(t,\tau)\,H^T(t,\tau)\,d\tau\right] \Rightarrow \exists\, u: \ x = \int_{t_m}^{t} H(t,\tau)\,u(\tau)\,d\tau =$$

$$= \int_{t_0}^{t} H(t,\tau)\,u(\tau)\,d\tau + \Phi(t, t_0)\int_{t_m}^{t_0} H(t_0,\tau)\,u(\tau)\,d\tau$$

hence $x \in \mathcal{R}(t)$.

Conversely, if x is strongly reachable, then $\forall t_0 < t$, $\exists x_0, u$:

$$x = \Phi(t, t_0)\,x_0 + \int_{t_0}^{t} H(t,\tau)\,u(\tau)\,d\tau \tag{13}$$

hence

$$x \in \mathcal{R}(t) \blacktriangleleft \tag{14}$$

It is well known that a state x is *unobservable* at time t if:

$$\Psi(\bar{t}, t)\, x = 0, \ \forall \bar{t} \geq t \tag{15}$$

and that the set of unobservable states is given by:

$$Q(t) = N\left[\int_t^{\infty} \Phi^T(\tau, t)\, C^T(\tau)\, C(\tau)\, \Phi(\tau, t)\, d\tau\right] \tag{16}$$

where $N[\cdot]$ denotes the null space.

It is now possible to state the following theorem:

THEOREM 1. (5,6) *represent an uniform system if and only if*:

$$\mathcal{R}(t) + Q(t) = \mathbf{R}^n, \ \forall t \in T. \tag{17}$$

PROOF: if (5, 6) represent an uniform system, (4) implies:

$$\forall t, \ \forall x, \ \forall u, \ \forall t_0 < t, \ \exists x_0, u_0:$$

$$\Psi(t_1, t)\, x + \int_t^{t_1} W(t_1, \tau)\, u(\tau)\, d\tau = \Psi(t_1, t_0)\, x_0 + \int_{t_0}^{t} W(t_1, \tau)\, u_0(\tau)\, d\tau +$$

$$+ \int_t^{t_1} W(t_1, \tau)\, u(\tau)\, d\tau, \quad \forall t_1 \geq t. \tag{18}$$

Hence, $\forall t, \ \forall x, \ \forall t_0 < t, \ \exists x_0, u_0$:

$$\Psi(t_1, t)\left[x - \Phi(t, t_0)\, x_0 + \int_{t_0}^{t} H(t, \tau)\, u_0(\tau)\, d\tau\right] = 0 \tag{19}$$

and putting $t_0 \leq t_m$ (17) follows.

Conversely suppose (17) is false; then

$$\exists t, \ \exists x: \ \forall x_r \in \mathcal{R}(t)$$

$$\Psi(t_1, t)\, (x - x_r) \neq 0, \ \forall t_1 \geq t \tag{20}$$

Hence

$$\exists t, \ \exists x: \ \forall x_0, u_0:$$

$$\Psi(t_1, t)\, x + \Psi(t_1, t) \left[\Phi(t, t_m)\, x_0 + \int_{t_m}^{t} H(t, \tau)\, u_0(\tau)\, d\tau \right] \tag{21}$$

and this proves that the system cannot be uniform. ◀

The second part of this proof shows that if (17) is not satisfied, there exists a state x at a time t giving rise only to outputs which are not restrictions of preceding outputs.

4. **Reversibility condition.**

It was shown in [6] (Theorem 2) that the special class of uniform systems represented by a linear input-output map is reversible if and only if a suitably defined function is surjective. That condition can be restated in a significant manner as a condition on the reduced state space at each time to have non increasing dimensions.

The same idea is extended in this section to the linear uniform systems, after giving a suitable definition of the reduced state space at each time.

Define (not uniquely) a subspace $B(t)$ as follows

$$\mathcal{R}(t) = \mathcal{R}(t) \cap Q(t) \oplus B(t) \tag{22}$$

Clearly all the states in $B(t)$ are strongly reachable and observable at time t; moreover it is easy to see that, defining the *set of free outputs at time* t as

$$L(t) = \{(0, y) \mid (0, y) \in S(t)\} \tag{23}$$

results

$$\dim B(t) = \dim L(t), \quad \forall t \tag{24}$$

Being all the states in $B(t)$ observable, there exists a linear one-to-one correspondence between $B(t)$ and $L(t)$, so that $B(t)$ can be taken as the reduced state space at time t. The reversibility condition can now be stated.

THEOREM 2. (5, 6) *represent a reversible system if and only if* (17) *holds and*

$$\dim B(t_1) \leq \dim B(t_0), \ \forall t_0, \ \forall t_1 \geq t_0 . \tag{25}$$

PROOF. Suppose that the system represented by (5,6) admits another representation of order m defined by $\widetilde{\Phi}$, $\widetilde{H}$, $\widetilde{\Psi}$, $\widetilde{W}$ matrices with

$\det \widetilde{\Phi}(t, t_0) \neq 0$ $\forall t$, $\forall t_0 \leq t$. Then (17) holds; moreover the set of free outputs at time t is given by

$$L(t_1) = \{(0, \widetilde{\Psi}(t, t_1)\, x_0) \mid x_0 \in \mathbf{R}^m\} \tag{26}$$

and at time $t_0 < t_1$

$$L(t_0)\,|_{T(t_0)} = \{(0, \widetilde{\Psi}(t, t_1)\, \widetilde{\Phi}(t_1, t_0)\, x_0) \mid x_0 \in \mathbf{R}^m\} \tag{27}$$

Being $\widetilde{\Phi}(t_1, t_0)$ invertible, it follows that

$$\dim L(t_1) \leq \dim L(t_0) \tag{28}$$

hence, by virtue of (24), (25) is satisfied.

Conversely, if (5, 6) are such that (17) holds, it follows that:

$$B(t) \oplus Q(t) = \mathbf{R}^n \tag{29}$$

and (25) implies:

$$\dim Q(t_1) \geq \dim Q(t_0), \quad \forall t_0, \quad \forall t_1 \geq t_0 \tag{30}$$

Hence, there exists a time t_q such that

$$\dim Q(t) = \dim Q(t_q) = n - l, \quad \forall t \leq t_q \tag{31}$$

Moreover, $\forall t_q$, $\forall t_1 \geq t_0$, $\forall x_1$, $\exists\, x_0, u$:

$$\Psi(t, t_1)\, x_1 = \Psi(t, t_1) \left[\Phi(t_1, t_0)\, x_0 + \int_{t_1}^{t_0} H(t_1, \tau)\, u(\tau)\, d\tau \right], \quad \forall t \geq t_1 \tag{32}$$

Choose a basis in $B(t_q)$ made up by vectors $x_1, \ldots, x_l$ and $\forall t$ define

$$y_i(t) = \begin{cases} \Psi(t, t_q)\, x_i \ \text{ if } \ t \geq t_q \\ \Psi(t, t_{-1})\, x_i' \ \text{ if } \ t < t_q \quad i = 1, \ldots, l \end{cases} \tag{33}$$

where $t_{-1} < t$ and x_i' is such that there exists an input u_i which satisfies the condition

$$\Psi(t, t_q) \left[\Phi(t_q, t_{-1})\, x_i' + \int_{t_1}^{t_q} H(t_q, \tau)\, u_i(\tau)\, d\tau \right] = \Psi(t, t_q)\, x_i, \quad \forall t \geq t_q \tag{34}$$

The outputs y_i are well-defined because:

i) (34) can be satisfied by virtue of (32);

ii) the choice of x_i' in (34) does not change the value of $y_i(t)$, because from the definition of t_q it follows that

$$\begin{aligned} &\Psi(t, t_{-1})\, x = \Psi(t, t_{-1})\, x', \quad \forall t \geq t_q \Rightarrow \\ &\Rightarrow \Psi(t, t_{-1})\, x = \Psi(t, t_{-1})\, x', \quad \forall t \geq t_{-1} \end{aligned} \tag{35}$$

iii) the same is true for the choice of t_{-1}.

Define the $q \times l$ matrix

$$\widetilde{\Psi}(t) = [y_1(t) \,\vdots\, \dots \,\vdots\, y_l(t)] \tag{36}$$

and remark that

$$\forall t_0, \ \forall x, \ \exists z \in \mathbf{R}^l:$$

$$\Psi(t, t_0)\, x = \widetilde{\Psi}(t)\, z, \quad \forall t \geq t_0 \tag{37}$$

Being

$$W(t, \tau) = \Psi(t, t_0)\, H(t_0, \tau), \ \forall t_0, \ \forall t \geq t_0, \ \text{ a. e. } \tau < t_0 \tag{38}$$

there exists an $l \times p$ matrix $\widetilde{H}(\tau)$ such that

$$W(t, \tau) = \widetilde{\Psi}(t)\, \widetilde{H}(\tau), \ \forall t, \text{ a. e. } \tau \leq t. \tag{39}$$

It is easy to see from (37), (39) that

$$z(t) = z_0 + \int_{t_0}^{t} \widetilde{H}(\tau)\, u(\tau)\, d\tau \tag{40}$$

$$y(t) = \widetilde{\Psi}(t)\, z_0 + \int_{t_0}^{t} \widetilde{\Psi}(t)\, \widetilde{H}(\tau)\, u(\tau)\, d\tau \tag{41}$$

constitutes a representation of the system. Being the transition matrix in (40) equal to the identity, the system is reversible. ◀

Remark that the representation (40), (41) is such that every state is strongly reachable and observable before t_q; it follows that its state space is reduced at each time before t_q. Hence the following result holds:

COROLLARY 1. *The representation* (40), (41) *is minimal.*

5. An example.

In this section an example is given to show an application of the previous theory. Consider the network in fig. 1 and the system obtained by taking the voltages u and y respectively as the input and the output. The switch is opened for $t<0$ and closed for $t\geq 0$. The system seems not to be uniform, because at $t=0$ a new state arises giving rise to new outputs. It is quite easy to understand when this fact does not destroy the uniformity of the system; however let us apply the theory of Section 3.

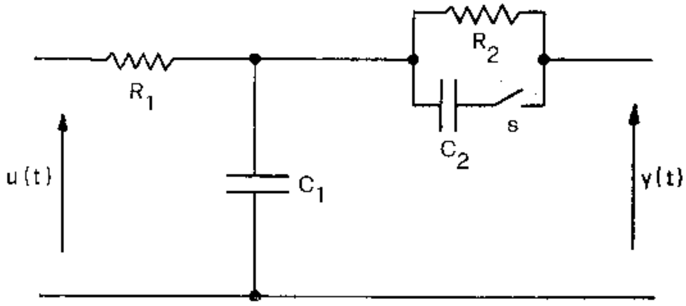

Fig. 1

The system can be represented by the following matrices:

$$\Phi(t, t_0)=\begin{cases} I & \text{if } t=t_0 \\ \begin{bmatrix} e^{-\frac{1}{R_1 C_1}(t-t_0)} & 0 \\ 0 & e^{-\frac{1}{R_2 C_2}(t-t_0)}\,\delta_{-1}(t_0) \end{bmatrix} & \text{if } t>t_0 \end{cases} \tag{42}$$

$$\Psi(t, t_0)=\left[\, e^{-\frac{1}{R_1 C_1}(t-t_0)} \quad e^{-\frac{1}{R_2 C_2}(t-t_0)}\,\delta_{-1}(t_0)\right] \tag{43}$$

$$H(t,\tau)=\begin{bmatrix} e^{-\frac{1}{R_1C_1}(t-t_0)} \\ 0 \end{bmatrix} \tag{44}$$

$$W(t,\tau)=e^{-\frac{1}{R_1C_1}(t-\tau)} \tag{45}$$

Applying (8) and (16) it is obtained that

$$\mathcal{R}(t)=\text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{46}$$

$$Q(t)=\begin{cases} \text{span}\begin{bmatrix} 0 \\ 1-\delta_{-1}(t) \end{bmatrix} & \text{if } R_1C_1\neq R_2C_2 \\ \text{span}\begin{bmatrix} \delta_{-1}(t) \\ -1 \end{bmatrix} & \text{if } R_1C_1=R_2C_2. \end{cases} \tag{47}$$

Hence the system is uniform if and only if $R_1C_1=R_2C_2$.

Moreover in this case it results

$$\dim B(t)=1, \quad \forall t \tag{48}$$

hence the system is reversible.

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