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ON THE REALIZATION OF VARIABLE-STRUCTURE INPUT-OUTPUT MAPS

This paper deals with Variable Structure Systems, that is, systems represented by differential equations being linear in the state but not in the input. For a fixed initial state such a system defines a Variable Structure Input-Output Map. The differential equation in the above mentioned initial state constitutes a realization of the Variable Structure Input-Output Map if and only if suitable conditions are satisfied. The set of all these realizations is characterized by constant matrices which are a generalization of the Hankel's matrix and restitute the latter in the linear case. Furthermore, minimality conditions for a realization of a Variable Structure Input-Output Map are given, and an algorithm for computing the minimal order is outlined.

I. INTRODUCTION

It is well known that a *System* is constituted by the collection of the sets of input-output function pairs which can be generated starting at each possible initial time [1, 2]. This paper deals with a special class of variable-structure systems, which are usually described by a differential equation of the following kind:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{F}[\boldsymbol{u}(t)]\boldsymbol{x}(t)+\boldsymbol{g}[\boldsymbol{u}(t)], \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t), \end{aligned} \tag{I-1}$$

where $\boldsymbol{x}(t)\in\mathscr{R}^n$, the input $\boldsymbol{u}(t)\in\mathscr{R}^p$, the output $\boldsymbol{y}(t)\in\mathscr{R}^q$ and the matrices have proper dimensions.

Eq. (I-1) allows the definition of a system in the above mentioned sense through the following steps. Assume as the time set $\mathscr{T}(t_0)$ an interval of the real line containing the lower extremum t_0, and choose an initial condition $\boldsymbol{x}_0$ for (I-1). Consider the input-output map $\boldsymbol{F}_v$: $(\mathscr{R}^p)^{\mathscr{T}(t_0)}\to(\mathscr{R}^q)^{\mathscr{T}(t_0)}$ which associates to the input function $\boldsymbol{u}$ the output $\boldsymbol{y}$ obtained by solving (I-1) with $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$,

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Put

$$\mathscr{S}(t_0) \triangleq \{(\boldsymbol{u}, \boldsymbol{y}): \boldsymbol{y} = \boldsymbol{F}_v(\boldsymbol{u}), \boldsymbol{u} \in (\mathscr{R}^p)^{\mathscr{T}(t_0)}\}. \tag{I-2}$$

Finally, denoting, $\forall t_1 \in \mathscr{T}(t_0)$ with

$$\mathscr{T}(t_1) = \{t: t \in \mathscr{T}(t_0) \,\&\, t \geqslant t_1\} \tag{I-3}$$

and

$$\mathscr{T}(t_1) = \{(\boldsymbol{u}|_{\mathscr{T}(t_1)}, \boldsymbol{y}|_{\mathscr{T}(t_1)}): (\boldsymbol{u}, \boldsymbol{y}) \in \mathscr{S}(t_0)\} \tag{I-4}$$

the *variable structure system* associated with eq. (I-1) and $\boldsymbol{x}_0$ is the set

$$\mathscr{S}_v \triangleq \{\mathscr{S}(t_1), t_1 \in \mathscr{T}(t_0)\}. \tag{I-5}$$

Clearly, $\mathscr{S}_v$ is completely characterized by the input-output map $\boldsymbol{F}_v$, and vice versa, while the set $(\boldsymbol{F}, \boldsymbol{g}, \boldsymbol{C}, \boldsymbol{x}_0)$ which defines $\boldsymbol{F}_v$ is generally not unique. It is useful to stress that the set $(\boldsymbol{F}, \boldsymbol{g}, \boldsymbol{C}, \boldsymbol{x}_0)$ does not constitute a state representation of $\mathscr{S}_v$, because the input-output pairs given by (I-1) at time $t_1 > t_0$ constitute, in general, a set richer than $\mathscr{S}(t_1)$. The set $(\boldsymbol{F}, \boldsymbol{g}, \boldsymbol{C}, \boldsymbol{x}_0)$ constitutes a *realization* of $\mathscr{S}_v$, that is a representation of the input-output map $\boldsymbol{F}_v$. This is well known for linear time-invariant systems, for which the realization problem was firstly formulated in [3] (in this case $\boldsymbol{x}_0$ is usually assumed to be the zero state), and for bilinear systems [4] (in this case $\boldsymbol{x}_0$ is not necessarily the zero-state [5]).

In this paper the realizations of a given $\mathscr{S}_v$ are studied and two main questions are solved in some sense:

(i) the description of the set of all the realizations of a given $\mathscr{S}_v$ (see section II);

(ii) the evaluation of the minimal order and the construction of minimal realization from a non-minimal one for a given $\mathscr{S}_v$ (see section III).

In both cases the results are a generalization of the ones which hold in the linear and bilinear cases; the comparisons are formulated in each section.

II. THE SET OF THE REALIZATIONS OF AN INPUT-OUTPUT MAP

The problem faced in this section is the following one: given two sets $(\boldsymbol{F}, \boldsymbol{g}, \boldsymbol{C}, \boldsymbol{x}_0)$ and $(\boldsymbol{F}' \boldsymbol{g}', \boldsymbol{C}', \boldsymbol{x}_0')$, find a criterion to establish whether or not they are realizations of the same $\mathscr{S}_v$. The first solution to this kind of problem, for the linear time-invariant case, was given by Kalman in [3], and was based on the Hankel matrix; namely, two triplets $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$, $(\boldsymbol{A}', \boldsymbol{B}', \boldsymbol{C}')$ realize the same zero-state input-output map if and only if the corresponding Hankel matrices coincide. This result has been generalized in [4, 5] for the bilinear case, through the definition of a new matrix very similar to the Hankel's one. Because linear and bilinear systems are special cases of variable structure systems, it seems natural to ask if there exists a matrix which is invariant over the set of all the realizations of an $\mathscr{S}_v$ and is reduced to the Hankel's one (or to its generalization) in the linear and bilinear cases.

A positive answer to this question is given in the present section:

First of all, let us introduce some useful notations as in [5]. Given r $n\times n$ matrices $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_r$ and two matrices $\boldsymbol{G}$ and $\boldsymbol{C}$, respectively $n\times s$, $q\times n$, define:

$$\bar{\boldsymbol{U}}_1 = \boldsymbol{G},\ \bar{\boldsymbol{U}}j = (\boldsymbol{F}_1\bar{\boldsymbol{U}}_{j-1} \vdots \ldots \vdots \boldsymbol{F}_r\bar{\boldsymbol{U}}_{j-1}),\ \boldsymbol{U}_j = (\bar{\boldsymbol{U}}_1 \vdots \ldots \vdots \bar{\boldsymbol{U}}_j)$$

$$\bar{\boldsymbol{V}}_1 = \boldsymbol{C}$$

$$\bar{\boldsymbol{V}}_j = \begin{bmatrix} \bar{\boldsymbol{V}}_{j-1}\boldsymbol{F}_1 \\ \cdots\cdots \\ \vdots \\ \cdots\cdots \\ \bar{\boldsymbol{V}}_{j-1}\boldsymbol{F}_r \end{bmatrix} \tag{II-1}$$

$$\boldsymbol{V}_j = \begin{bmatrix} \bar{\boldsymbol{V}}_1 \\ \cdots\cdots \\ \vdots \\ \cdots\cdots \\ \bar{\boldsymbol{V}}_j \end{bmatrix}$$

and

$$\boldsymbol{H}_j = \boldsymbol{V}_j\boldsymbol{U}_j. \tag{II-2}$$

A first result can now be stated for the particular case of zero initial conditions; then the general result will be given.

THEOREM 1

Two sets $(\boldsymbol{F}, \boldsymbol{g}, \boldsymbol{C}, \boldsymbol{O})$ *and* $(\boldsymbol{F}', \boldsymbol{g}', \boldsymbol{C}', \boldsymbol{O}')$, *of the respective orders* n *and* n', *are realizations of the same* $\mathscr{S}_v$ *if and only if*

$$\boldsymbol{H}_{\bar{n}} = \boldsymbol{H}'_{\bar{n}} \tag{II-3}$$

where

$$\bar{n} = \max(n, n'), \tag{II-4}$$

$$\boldsymbol{G} = (\boldsymbol{g}_1 \vdots \ldots \vdots \boldsymbol{g}_r), \tag{II-5}$$

$$\boldsymbol{G}' = (\boldsymbol{g}'_1 \vdots \ldots \vdots \boldsymbol{g}'_r), \tag{II-6}$$

and $\boldsymbol{g}_i, \boldsymbol{g}'_i, \boldsymbol{F}_i, \boldsymbol{F}'_i$ *are values of* $\boldsymbol{g}(\cdot), \boldsymbol{g}'(\cdot), \boldsymbol{F}(\cdot), \boldsymbol{F}'(\cdot)$ *such that*

$$\exists r \leqslant n^2 + n'^2 + n + n' :\ \forall \boldsymbol{u} \in \mathscr{R}^p \exists \lambda_1, \ldots, \lambda_r : \tag{II-7}$$

$$\big(\boldsymbol{F}(\boldsymbol{u}), \boldsymbol{F}'(\boldsymbol{u}), \boldsymbol{g}(\boldsymbol{u}), \boldsymbol{g}'(\boldsymbol{u})\big) = \sum_{i=1}^{r} \lambda_i(\boldsymbol{F}_i, \boldsymbol{F}'_i, \boldsymbol{g}_i, \boldsymbol{g}'_i).$$

Proof

As regards the necessity, choose an input as in fig. 1 where the values $u_1, \dots, u_r$ are those utilized in (II-7) and the instants of time $t_1, \dots, t_r$ can be arbitrarily chosen; then

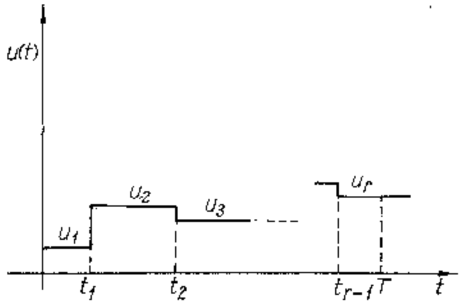

Fig. 1.

the corresponding outputs in the zero initial conditions are given by:

$$\begin{aligned}\boldsymbol{y}(T) &= \boldsymbol{C}\exp\left(\boldsymbol{F}_r(T-t_{r-1})\right)\dots\exp\left(\boldsymbol{F}_2(t_2-t_1)\right)\int_{t_0}^{t_1}\exp\left(\boldsymbol{F}_1(t_1-\tau)\right)\boldsymbol{g}_1\,\mathrm{d}\tau+\\ &+\dots+\boldsymbol{C}\exp\left(\boldsymbol{F}_r(T-t_{r-1})\right)\int_{t_{r-2}}^{t_{r-1}}\exp\left(\boldsymbol{F}_{r-1}(t_{r-1}-\tau)\right)\boldsymbol{g}_{r-1}\,\mathrm{d}\tau+\\ &+\boldsymbol{C}\int_{t_{r-1}}^{T}\exp\left(\boldsymbol{F}_r(T-\tau)\right)\boldsymbol{g}_r\,\mathrm{d}\tau,\end{aligned} \tag{II-8}$$

and $\boldsymbol{y}'(T)$ has the same form.

By replacing the exponential functions in (II-8) with the corresponding series-developments it is easy to verify that:

$$\boldsymbol{y}(T) = \boldsymbol{y}'(T) \quad \forall T\in[t_0, \infty] \tag{II-9}$$

implies:

$$\begin{gathered}\boldsymbol{C}\boldsymbol{F}_{i_1}^{k_1}\dots\boldsymbol{F}_{i_r}^{k_r}\boldsymbol{g}_s = \boldsymbol{C}'\boldsymbol{F}_{i_1}'^{k_1}\dots\boldsymbol{F}_{i_r}'^{k_r}\boldsymbol{g}_s',\\ \forall i_1, \dots, i_r = 1, \dots, r,\\ \forall k_1, \dots, k_r = 1, \dots, k-1,\\ \forall s = 1, \dots, r.\end{gathered} \tag{II-10}$$

Hence the necessity is proved.

Conversely, writing the output in the form of the Neumann series:

$$\boldsymbol{y}(T) = \boldsymbol{C}\int_{t_0}^{T}\left[\boldsymbol{I}+\int_{\tau}^{T}\boldsymbol{F}(\boldsymbol{u}(\tau_1))\,\mathrm{d}\tau_1+\int_{\tau}^{T}\boldsymbol{F}(\boldsymbol{u}(\tau_1))\int_{\tau}^{\tau_1}\boldsymbol{F}(\boldsymbol{u}(\tau_2))\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1+\dots\right]\boldsymbol{g}(\boldsymbol{u}(\tau))\,\mathrm{d}\tau \tag{II-11}$$

it follows that:

$$\begin{aligned}\boldsymbol{y}(T)-\boldsymbol{y}'(T) &= \int_{t_0}^{T}\left[\boldsymbol{C}\boldsymbol{g}(\boldsymbol{u}(\tau))-\boldsymbol{C}'\boldsymbol{g}'(\boldsymbol{u}(\tau))\right)\mathrm{d}\tau+\\ &+\int_{t_0}^{T}\int_{\tau}^{T}\left[\boldsymbol{C}\boldsymbol{F}(\boldsymbol{u}(\tau_1))\boldsymbol{g}(\boldsymbol{u}(\tau))-\boldsymbol{C}'\boldsymbol{F}'(\boldsymbol{u}(\tau_1))\boldsymbol{g}'(\boldsymbol{u}(\tau))\right]\mathrm{d}\tau_1\,\mathrm{d}\tau+\dots\end{aligned} \tag{II-12}$$

Applying the generalized-mean theorem to (II-12) we have:

$$\forall T \in [t_0, \infty), \exists \xi_i, \quad i = 1, 2, 3, \ldots:$$

$$\boldsymbol{y}(T) - \boldsymbol{y}'(T) = [\boldsymbol{C}\boldsymbol{g}(\xi_1) - \boldsymbol{C}'\boldsymbol{g}'(\xi_1)](T - t_0) + \\ + [\boldsymbol{C}\boldsymbol{F}(\xi_1)\boldsymbol{g}(\xi_2) - \boldsymbol{C}'\boldsymbol{F}'(\xi_1)\boldsymbol{g}'(\xi_2)]\frac{(T-t_0)^2}{2} + \ldots \tag{II-13}$$

(II-7) and (II-3) imply:

$$\boldsymbol{C}\boldsymbol{g}(\xi_1) - \boldsymbol{C}'\boldsymbol{g}'(\xi_1) = \sum_{i=1}^{r} \lambda_i(\boldsymbol{C}\boldsymbol{g}_i - \boldsymbol{C}'\boldsymbol{g}'_i) = \boldsymbol{0} \tag{II-14}$$

and

$$\boldsymbol{C}\boldsymbol{F}(\xi_1)\boldsymbol{g}(\xi_2) - \boldsymbol{C}'\boldsymbol{F}'(\xi_1)\boldsymbol{g}'(\xi_2) = \sum_{j=1}^{r}\sum_{i=1}^{r} \gamma_j\lambda_i[\boldsymbol{C}\boldsymbol{F}_j\boldsymbol{g}_i - \boldsymbol{C}'\boldsymbol{F}'_j\boldsymbol{g}'_i] = \boldsymbol{0} \tag{II-15}$$

and so on for all the terms of the r.h.s. of (II-13); this completes the proof.

Consider the sets $(\boldsymbol{F}, \boldsymbol{g}, \boldsymbol{C}, \boldsymbol{x}_0)$ and $(\boldsymbol{F}', \boldsymbol{g}', \boldsymbol{C}', \boldsymbol{x}'_0)$ that individuate two differential equations of the kind (I-1) with $\boldsymbol{x}(t) \in \mathscr{R}^n$ and $\boldsymbol{x}'(t) \in \mathscr{R}^{n'}$.

By putting:

$$\boldsymbol{p}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_0, \tag{II-16}$$

$$\boldsymbol{p}'(t) = \boldsymbol{x}'(t) - \boldsymbol{x}'_0 \tag{II-17}$$

it follows that:

$$\begin{aligned} \dot{\boldsymbol{p}}(t) &= \boldsymbol{F}(\boldsymbol{u}(t))\boldsymbol{p}(t) + [\boldsymbol{F}(\boldsymbol{u}(t))\boldsymbol{x}_0 + \boldsymbol{g}(\boldsymbol{u}(t))], \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{p}(t) + \boldsymbol{C}\boldsymbol{x}_0, \\ \boldsymbol{p}(t_0) &= \boldsymbol{0}, \end{aligned} \tag{II-18}$$

and:

$$\begin{aligned} \dot{\boldsymbol{p}}'(t) &= \boldsymbol{F}'(\boldsymbol{u}(t))\boldsymbol{p}'(t) + [\boldsymbol{F}'(\boldsymbol{u}(t))\boldsymbol{x}'_0 + \boldsymbol{g}'(\boldsymbol{u}(t))], \\ \boldsymbol{y}'(t) &= \boldsymbol{C}'\boldsymbol{p}'(t) + \boldsymbol{C}'\boldsymbol{x}'_0, \\ \boldsymbol{p}'(t_0) &= \boldsymbol{0}. \end{aligned} \tag{II-19}$$

It is now possible to understand how to extend the result of Theorem 1 to the general case.

COROLLARY 1

Two sets $(\boldsymbol{F}, \boldsymbol{g}, \boldsymbol{C}, \boldsymbol{x}_0)$ *and* $(\boldsymbol{F}', \boldsymbol{g}', \boldsymbol{C}', \boldsymbol{x}'_0)$, *of orders* n *and* n' *respectively, are realizations of the same* $\mathscr{S}_v$ *if and only if:*

$$\boldsymbol{C}\boldsymbol{x}_0 = \boldsymbol{C}'\boldsymbol{x}'_0, \tag{II-20}$$

$$\boldsymbol{H}_{\bar{n}} = \boldsymbol{H}'_{\bar{n}}, \tag{II-21}$$

where:

$$\bar{n} = \max(n, n') \tag{II-22}$$

$$\boldsymbol{G} \triangleq (\boldsymbol{F}_1\boldsymbol{x}_0 + \boldsymbol{g}_1 \,\vdots \ldots \vdots\, \boldsymbol{F}_r\boldsymbol{x}_0 + \boldsymbol{g}_r), \tag{II-23}$$

$$\boldsymbol{G}' \triangleq (\boldsymbol{F}'_1\boldsymbol{x}'_0 + \boldsymbol{g}'_1 \,\vdots \ldots \vdots\, \boldsymbol{F}'_r\boldsymbol{x}'_0 + \boldsymbol{g}'_r), \tag{II-24}$$

and $F_i x_0 + g_i$, $F'_i x'_0 + g'_i$, F_i, F'_i *are values of* $F(\cdot)x_0 + g(\cdot)$, $F'(\cdot)x'_0 + g'(\cdot)$, $F(\cdot), F'(\cdot)$ *such that*:

$$\exists r \leqslant n^2 + n'^2 + n + n' : \ \forall u \in \mathscr{R}^p \exists \lambda_1, \ldots, \lambda_r :$$

$$\big(F(u), F'(u), F(u)x_0 + g(u), F'(u)x'_0 + g'(u)\big) \tag{II-25}$$

$$= \sum_{i=1}^{r} \lambda_i (F_i, F'_i, F_i x_0 + g_i, F'_i x'_0 + g'_i).$$

REMARK 1

The result expressed in Corollary 1, generalizes the well known results for the linear and bilinear cases. On the other hand the reader will have no difficulties to obtain the results in [3, 4, 5] from Corollary 1.

III. MINIMAL REALIZATIONS

It is well known that the minimal order of the realizations of a linear input-output map is given by the rank of the Hankel matrix [3]; this result has been generalized to the bilinear case in [4, 5]. Moreover, in both cases the minimality is characterized by the properties of reachability and observability of the state-space. In [6] this kind of properties has been studied for the variable structure systems, and the state-space has been decomposed into four linear manifolds. On these grounds it has been shown how to find a transformation such that the matrices of the system assume the following form:

$$F\big(u(t)\big) = \begin{pmatrix} F_{aa} & F_{ab} & F_{ac} & F_{ad} \\ 0 & F_{bb} & 0 & F_{bd} \\ 0 & 0 & F_{cc} & F_{cd} \\ 0 & 0 & 0 & F_{dd} \end{pmatrix}$$

$$F\big(u(t)\big)x_0 + g\big(u(t)\big) = \begin{pmatrix} (Fx_0 + g)_a \\ (Fx_0 + g)_b \\ 0 \\ 0 \end{pmatrix}$$

(III-1)

$$C = (0 \ C_b \ 0 \ C_d), \quad x_0 = \begin{pmatrix} x_{0a} \\ x_{0b} \\ x_{0c} \\ x_{0d} \end{pmatrix}$$

From (III-1) it follows that the input-output function $\boldsymbol{F}_v$ can be constructed from the following equations:

$$\dot{\boldsymbol{x}}_b(t) = \boldsymbol{F}_{bb}\big(\boldsymbol{u}(t)\big)\boldsymbol{x}_b(t) + \big[\boldsymbol{F}_{bd}\big(\boldsymbol{u}(t)\big)\boldsymbol{x}_{0d} + \boldsymbol{g}_b\big(\boldsymbol{u}(t)\big)\big],$$
$$\boldsymbol{y}(t) = \boldsymbol{C}_b\boldsymbol{x}_b(t) + \boldsymbol{C}_d\boldsymbol{x}_{0d}, \tag{III-2}$$
$$\boldsymbol{x}(t_0) = \boldsymbol{x}_{0b}.$$

If

$$\exists\, \boldsymbol{k}:\ \boldsymbol{C}_b\cdot\boldsymbol{k} = \boldsymbol{C}_d\cdot\boldsymbol{x}_{0d} \tag{III-3}$$

it is possible to obtain from (III-1) an equation of the form (I-1) by putting:

$$\boldsymbol{z}(t) = \boldsymbol{x}_b(t) - \boldsymbol{k}. \tag{III-4}$$

If (III-3) is not true, it seems natural to consider (III-2) as a realization of $\mathscr{S}_v$, the only difference being a constant term added to the output. With this assumption the following theorem can be stated:

THEOREM 2
The order of the minimal realizations of a given $\mathscr{S}_v$ equals the dimension of the least linear manifold containing all the states which are reachable and distinguisheable from $\boldsymbol{x}_0$.

PROOF
The previous reasoning shows that every minimal realization is such that its state space is spanned by states reachable and distinguisheable from $\boldsymbol{x}_0$.

Conversely, if a realization of order n is such that its state space is reachable and distinguisheable from $\boldsymbol{x}_0$, it follows that the matrices $\boldsymbol{V}_n$ and $\boldsymbol{U}_n$ which factorize the matrix $\boldsymbol{H}_n$ of Corollary 1 are of full rank; hence, from Sylvester's inequality:

$$\text{rank}\boldsymbol{H}_n = n \tag{III-5}$$

this implies that a realization of order lower than n cannot exist.

COROLLARY 2
The order of the minimal realizations of a given $\mathscr{S}_v$ is equal to the rank of $\boldsymbol{H}_n$.

IV. CONCLUSIONS

The set of all the realizations of a variable structure input-output map has been characterized through a suitable generalization of the Hankel matrix.

It would be of interest to find the structure of this set, particularly to investigate the role played by the algebraic equivalence in this class of systems.

The results about minimality can be used to find algorithms for reduction and for evaluation of the minimal order.

РЕАЛИЗАЦИЯ СХЕМ С ПЕРЕМЕННОЙ СТРУКТУРОЙ

Темой работы являются системы с переменной структурой, т.е. системы, представленные дифференциальными уравнениями, которые, будучи линейными по состоянию оказываются нелинейными на входе. Для определенного начального состояния такая система устанавливает схемы изменения структуры вход-выход. Дифференциальные уравнения в указанном начальном состоянии реализуют схему изменения структуры вход-выход в том, и только в том случае, если удовлетворяются соответствующие условия. Множество всех этих реализаций характеризуется при помощи постоянных матриц, которые являются обобщением матрицы Генкеля и воспроизводятся в случае линейности. Кроме того, приведены условия минимальности для реализации плана переменной структуры вход-выход и составлен алгоритм вычисления минимального порядка.

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REFERENCES

[1] Zadeh L. A., [in:] Zadeh & Polak, *Systems Theory*, Mc Graw Hill, 1969.

[2] Ruberti A., Isidori A., *Systems Theory*, (In Italian), Siderea, Rome 1974.

[3] Kalman R. E., *Mathematical description of linear dynamical systems*, SIAM J. Control, Vol. 1 pp. 152–192, 1963.

[4] d'Alessandro P., Isidori A., Ruberti A., *Realization and structure theory of bilinear dynamical systems*, SIAM J. Control, Vol. 12, 1974.

[5] Isidori A., Ruberti A., *Realization Theory of Bilinear Systems* [In:] Mayne, Brockett, *Geometric Methods in Systems Theory*, Reidel 1973.

[6] Gori-Giorgi C., Monaco S., *Structure Theory of Affine State Systems*, Submitted to J. Franklin Institute, 1975.

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