Structure Theory of State-Affine Systems

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ABSTRACT: *In this paper the structure theory of the state space is developed for a class of state-affine systems.*

On the basis of the properties of reachability and indistinguishability with respect to a given state, the state space decomposition is operated and the input–output behaviour is characterized.

I. Introduction

This paper deals with time-invariant state-affine systems represented by

$$\dot{x}(t) = F\,|u(t)|\,x(t) + g\,|u(t)| \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x(t) \in R^n$, $u(t) \in R^p$, $u(\cdot) \in U$, the space of admissible input functions, $y(t) \in R^q$, and $F(\cdot)$, $g(\cdot)$ are continuous functions. Recently this kind of systems has been widely investigated because of its usefulness in modelling and control theory.

Two main lines can be found in the literature: (i) a structure and realization theory for the special case of bilinear systems, with some interest in optimal control (**1–4**); (ii) a control theory for applications to special areas (**5, 6**), and for the analysis of special variable structure systems.

In this paper, a state-space structure theory for systems Eqs. (1) and (2) is presented. The role of the structure theory for analysis and synthesis of linear control systems is well known; its relevance lies in the connection between reachability and pole assignability, observability and state reconstruction, minimality and simulation. Furthermore, the structure theory is of wide relevance from a system–theoretic viewpoint, so that it seems justified to extend the structure theory to a wider classes of systems.

In Section II the definitions of reachability and undistinguishability with respect to an arbitrary state are given. In Sections III and IV two sets characterizing the reachable states and the indistinguishable states from a given initial state are found; lastly, in Section V a state space decomposition is presented and the input–output map from a given initial state is investigated.

II. Reachability and Indistinguishability

The role of reachability and indistinguishability from the origin in linear system theory is well known (**7**). The state space decomposition based on these properties, as proposed by Kalman, allows the decomposition of the whole system into four sub-systems only one of which characterizes the input–output map from the initial-zero-state. In the linear case the output of the system is the sum of a term which represents the output from an initial state $x_0 \neq 0$ with $u(t) = 0$, and a term which represents the input–output map from an initial-zero-state, the latter being the more meaningful.

Moreover, the impulse response in the zero-state constitutes a complete model of the reachable and observable subsystem; this is an important *a posteriori* justification of the whole theory.

All these motivations, based on the linearity, fail when the system is non-linear; for example, the following bilinear system:

$$\dot{x}(t) = Nx(t)u(t) \tag{3}$$

shows that the zero-state response can be meaningless (**8**).

To overcome such difficulties, in (**8**) the reachability property was defined with respect to an equilibrium state. Such an approach is well justified but not the most general; furthermore, on these bases the state space was decomposed and the input–output map was characterized if the origin of the state space belongs to the span of the states which are reachable from the equilibrium state. In the present paper the state space is decomposed and the input–output map is characterized with respect to an arbitrary initial state. To this purpose the definitions of reachability and indistinguishability are stated with respect to an arbitrary initial state.

Definition 1. A state x of the system (1), is reachable from x_0 if there exists an input which transfers the state x_0 into x in a finite interval of time.

Definition 2. A state x of the system (1), (2) is indistinguishable from x_0 if the outputs from x and from x_0 coincide for any input.

Obviously these definitions coincide with the usual ones when $x_0 = 0$.

III. Reachability

First, the definitions of some useful invariant sub-spaces will be recalled. If $F_1, F_2, \ldots, F_r$ are $n \times n$ matrices and G is an $n \times s$ matrix,

$$\text{gen}_{F_1,\ldots,F_r}(G) \quad \text{and} \quad \overline{\text{gen}}_{F_1,\ldots,F_r}(G^T) \tag{4}$$

are respectively the smallest sub-space of R^n invariant under $F_1, \ldots, F_r$ containing the range space of G and the largest sub-space of R^n invariant under $F_1, \ldots, F_r$ contained in the null space of G^T.

The properties of these sub-spaces and the algorithms for their construction are given in Section 3 of (**8**), to which the reader is referred.

Starting from the reachability property, note that the set of the states which are reachable from a given initial state is not, in general, a linear space.

Nevertheless in order to decompose the state space it seems natural to look for the smallest linear manifold containing the reachable set from an arbitrary initial state, in order to obtain a set with some linear structure.

To obtain this goal the following steps are necessary:

(1) It will be shown (Lemma 1) that $\bar{x}$ is reachable from x_0 for the system (1), if and only if $\bar{x}-x_0$ is reachable from the origin for another suitable variable-structure system;
(2) The span of the states which are reachable from the origin for a variable-structure system will be found (Theorem I);
(3) On the bases of these results the smallest linear manifold containing the states which are reachable from x_0 for the system (1), will be found.

Lemma 1. The state $\bar{x}$ is reachable from x_0 for the system (1), if and only if $\bar{x}-x_0$ is reachable from the origin for the system:

$$\dot{z}(t)=F\,|u(t)|\,z(t)+|F(u(t))x_0+g(u(t))|. \tag{5}$$

Proof: (1) yields to (5) by substituting $x=z+x_0$.

If $\bar{x}-x_0$ is reachable from the origin of the system (5), then there exists an input $\bar{u}$ and an instant of time T, such that

$$\begin{aligned}\bar{x}-x_0&=\int_0^T \Phi_{\bar{u}}(t,\tau)\,|F|\,\bar{u}(\tau)\,|x_0+g(\bar{u}(\tau))|\,\mathrm{d}\tau\\ &=\int_0^T \Phi_{\bar{u}}(t,\tau)F\,|\bar{u}(\tau)|\,\mathrm{d}\tau\cdot x_0+\int_0^T \Phi_u(t,\tau)g(u(\tau))\,\mathrm{d}\tau\end{aligned} \tag{6}$$

where $\Phi_u(t,\tau)$ is the solution of

$$\dot{X}(t)=F\,|u(t)|\,X(t),\quad X(\tau)=I \tag{7}$$

Eq. (7) then implies

$$-\frac{\partial}{\partial t}\Phi_{\bar{u}}(t,\tau)=\Phi_{\bar{u}}(\tau,t)\cdot F\,|\bar{u}(t)|. \tag{8}$$

From Eqs. (6) and (8), it follows that

$$\begin{aligned}\bar{x}-x_0&=\Phi_{\bar{u}}(T,0)\cdot\int_0^T \Phi_{\bar{u}}(0,\tau)F\,|\bar{u}(\tau)|\,\mathrm{d}\tau\cdot x_0+\int_0^T \Phi_{\bar{u}}(t,\tau)g(u(\tau))\,\mathrm{d}\tau\\ &=\Phi_{\bar{u}}(T,0)\,|-\Phi_{\bar{u}}(0,T)+I|\,x_0+\int_0^T \Phi_{\bar{u}}(t,\tau)g(u(\tau))\,\mathrm{d}\tau.\end{aligned} \tag{9}$$

Hence $\bar{x}$ is reachable from x_0 for the system (1); the converse can be easily proved in the same manner.

Recall that a sub-space $S \subset R^n$ is invariant under $F(\cdot)$ if $x \in S \rightarrow F(u) \cdot x \in S$, $\forall u \in R^p$, and that the range of $g(\cdot)$ is defined by

$$R(g) = \{x \mid \exists u \in R^p: \quad x = g(u)\}. \tag{10}$$

The following theorem can now be stated:

Theorem I

The span $P(0)$ of the states reachable from the origin for the system (1) is the smallest sub-space invariant under $F(\cdot)$ containing $R(g)$.

Proof: Note that, $\forall u \in R^p$, $F(u)$ and $g(u)$ are elements belonging respectively to the vector spaces $R^{n\times n}$ and R^n. Hence it is always possible to find input values $u_1, \ldots, u_r$, $r \leq n^2+n$ such that

$$\begin{aligned} &\forall u \in R^p, \exists \lambda_1, \ldots, \lambda_r \in R: \quad F(u) = \sum_{i=1}^{r} \lambda_i F(u_i) \\ &\forall u \in R^p, \exists \gamma_1, \ldots, \gamma_r \in R: \quad g(u) = \sum_{i=1}^{r} \gamma_i g(u_i). \end{aligned} \tag{11}$$

By putting $F_i \triangleq F(u_i)$, $g_i \triangleq g(u_i)$, $i = 1, \ldots, r$, the theorem will be proved if it is proven that

$$P(0) = \text{gen}_{F_1,\ldots,F_r}(g_i \vdots \ldots \vdots g_r). \tag{12}$$

First let us observe that

$$\exists T, \bar{u} : x = \int_0^T \Phi_u(t, \tau) g(\bar{u}(\tau))\, d\tau \rightarrow x \in P(0) \tag{13}$$

hence:

$$\gamma \in P^{\perp}(0) \rightarrow \forall T \in R, \forall \bar{u} \in U, \gamma^T \int_0^T \Phi_u(t, \tau) g(u(\tau))\, d\tau = 0. \tag{14}$$

Choose an input as in Fig. 1 where the values $u_1, \ldots, u_r$ are those utilized in (11) and the instants of time $t_1, \ldots, t_r$ can be arbitrarily chosen; then the motion of the state starting from the origin is given by

$$\begin{aligned} x(T) = {} & \exp(F_r(T - t_{r-1})) \exp(F_{r-1}(t_{r-1} - t_{r-2})) \ldots \exp(F_r(t_2 - t_1)) \\ & \times \int_0^{t_1} \exp(F_1(t_1 - \tau)) g_1\, d\tau + \ldots + \exp(F_r(T - t_{r-1})) \\ & \times \int_{t_{r-2}}^{t_{r-1}} \exp(F_{r-1}(t_{r-1} - \tau)) g_{r-1}\, d\tau + \int_{t_{r-1}}^{T} \exp(F_r(T - \tau)) g_r\, d\tau. \end{aligned} \tag{15}$$

By replacing the exponential functions in Eq. (15) with the corresponding series developments, it is easy to verify that (14) implies

$$\gamma \in P^{\perp}(0) \rightarrow \gamma^T F_{i_1}^{k_1} \quad F_{i_2}^{k_2} \ldots F_{i_r}^{k_r} g_k = 0 \tag{16}$$

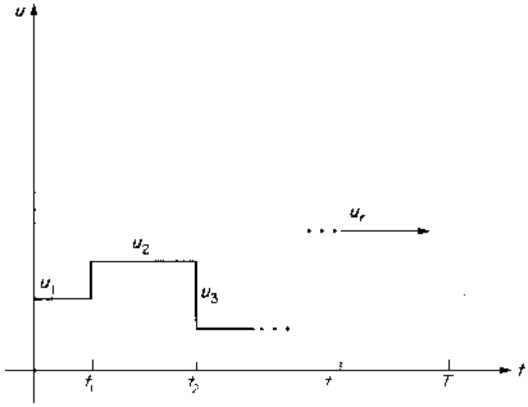

FIG. 1.

$$\forall i_1, \ldots, i_r = 1, \ldots, r$$
$$\forall k_1, \ldots, k_r = 1, \ldots, k-1$$
$$\forall k = 1, \ldots, r;$$

hence:

$$\gamma \in P^{\perp}(0) \rightarrow \gamma \in \mathrm{gen}_{F_1,\ldots,F_r}(g_1 \vdots \ldots \vdots g_r) \tag{17}$$

that is,

$$P(0) \supseteq \mathrm{gen}_{F_1,\ldots,F_r}(g_1 \vdots \ldots \vdots g_r). \tag{18}$$

Conversely, Eq. (11) implies that

$$\gamma \in \mathrm{gen}^{\perp}_{F_1,\ldots,F_r}(g_1 \vdots \ldots \vdots g_r) \rightarrow \gamma^T F(\xi_1) F(\xi_2) \ldots F(\xi_k) g(\xi) = 0 \qquad \forall k \in N, \quad \forall \xi_i \in R^p, \quad \forall \xi \in R^p. \tag{19}$$

By re-writing the l.h.s. of (13) in the form of the Neumann series, we have

$$x = \left(I + \int_0^T F(\bar{u}(\tau_1))\, \mathrm{d}\tau_1 + \int_0^T F(\bar{u}(\tau_1)) \int_0^{T_1} F(\bar{u}(\tau_2))\, \mathrm{d}\tau_2\, \mathrm{d}\tau_1 + \ldots\right) \times \left| \int_0^T \left(I + \int_\tau^0 F(\bar{u}(\tau_1))\, \mathrm{d}\tau_1 + \int_\tau^0 F(\bar{u}(\tau_1)) \int_\tau^{\tau_1} F(\bar{u}(\tau_2))\, \mathrm{d}\tau_2\, \mathrm{d}\tau_1 + \ldots\right) g(\bar{u}(\tau))\, \mathrm{d}\tau \right|. \tag{20}$$

By applying the generalized mean theorem to (20), we obtain

$\forall T \in R, \exists \xi_i, \zeta_i, i = 1, 2, 3, \ldots:$

$$x = \left(I + F(\xi_1)T + F(\xi_2)F(\xi_3)\frac{T^2}{2} + \ldots\right) \cdot \left(g(\zeta_1)T - F(\zeta_2)g(\zeta_3)\frac{T^2}{2} + \ldots\right) \tag{21}$$

Eqs. (19) and (21) then imply

$$\gamma \in \mathrm{gen}^{\perp}_{F_1,\ldots,f_r}(g_1 \vdots \ldots \vdots g_r)(\gamma \in P^{\perp}(0) \tag{22}$$

and this completes the proof.

Corollary 1. The smallest linear manifold $P(x_0)$ containing the set of the states which are reachable from x_0 for the system (1) is given by

$$P(x_0) = \{x_0 + z \mid z \in I\} \tag{23}$$

where I is the smallest sub-space invariant under $F(\cdot)$ and containing the range of $F(\cdot)x_0 + g(\cdot)$.

Proof: Choose, as in the proof of Theorem 1, input values $u_1, \ldots, u_r$ such that the first of (11) holds and the following condition is satisfied:

$$\forall u \in R^p, \exists \gamma_1, \ldots, \gamma_r \in R : F(u)x_0 + g(u) = \sum_{i=1}^{r} \gamma_i (F_i x_0 + g_i). \tag{24}$$

With these positions (23) is equivalent to

$$P(x_0) = \{x_0 + z \mid z \in \mathrm{gen}_{F_1,\ldots,F_r}(F_1 x_0 + g_1 \vdots \ldots \vdots F_r x_0 + g_r)\} \tag{25}$$

thus the proof is obvious.

Remark 1. The result of corollary 1 allows us to find and extend the well known results for special classes of state-affine systems, i.e.

(i) For a linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{26}$$

the input values $u_1, \ldots, u_r$ required in the first of (11) and in (24) can be chosen as follows:

$$u_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} i\text{th position} \quad i = 1, \ldots, p; \quad u_{p+1} = 0.$$

This allows us to obtain the well known expression:

$$P(0) = \mathrm{gen}_A(B).$$

Moreover, by writing $B = (b_1 \vdots \ldots \vdots b_p)$, we obtain

$$P(x_0) = \mathrm{gen}_A(Ax_0 + b_1 \vdots \ldots \vdots Ax_0 + b_p). \tag{27}$$

(ii) For a time invariant bilinear system:

$$\dot{x}(t) = \left(A + \sum_{k=1}^{p} N_k u_k(t)\right)x(t) + Bu(t) \tag{28}$$

the same choice of the input values as that of the linear case allows us to obtain (**8**),

$$P(0) = \mathrm{gen}_{A, A+N_1, \ldots, A+N_p}(B) = \mathrm{gen}_{A, N_1, \ldots, N_p}(B) \tag{29}$$

and in general,

$$P(x_0) = \mathrm{gen}_{A, N_1, \ldots, N_p}((A+N_1)x_0 + b_1 \vdots \ldots \vdots (A+N_p)x_0 + b_p); \tag{30}$$

which is an extension of an expression found in (**8**).

IV. Indistinguishability

It is easy to verify that the set of states indistinguishable from a fixed x_0 is a linear manifold; therefore it seems natural to look for this linear manifold without embedding it in a more suitable structure. To achieve this result, the steps will be similar to those of the preceding section; hence the procedure will only be outlined.

Lemma 2. The state $\bar{x}$ is indistinguishable from x_0 for the system (1), (2) if and only if $\bar{x} - x_0$ is indistinguishable from the origin for the system represented by (5) and

$$y(t) = Cz(t) + Cx_0 \tag{31}$$

Theorem II

The set $Q(0)$ of the states indistinguishable form the origin for the system (1), (2) is the largest sub-space of R^n invariant under $F(\cdot)$ and contained in the null space of C.

Proof: Note that, on the bases of the above positions, the theorem states

$$Q(0) = \overline{\mathrm{gen}}_{F_1, \ldots, F_r}(C) \tag{32}$$

where $F_1, \ldots, F_r$ satisfy (11).

To prove that a state indistinguishable from the origin belongs to the r.h.s. of (32), it is sufficient to consider the input function in Fig. 1 by repeating the same considerations of the preceding section. Conversely, if a state belongs to the r.h.s. of (32) it is necessarily indistinguishable as it is easy to verify by (21).

Remark 2. The result given by Theorem II remains unchanged if a constant term is added to the r.h.s. of (2); therefore such a result can be used in Lemma 2 also if (5), (31) is not a state-affine system.

Corollary 2. The set $Q(x_0)$ of the states indistinguishable from x_0 for the system (1), (2) is given by

$$Q(x_0) = \{x_0 + z \mid z \in L\} \tag{33}$$

where L is the largest sub-space invariant under $F(\cdot)$ and contained in the null space of C.

On the basis of the results stated in this section it is easy to complete examples (i), (ii) and (iii) in Remark 1.

Remark 3. Corollary 2 gives the set of states indistinguishable from x_0, which is a linear manifold, while Corollary 1 gives the smallest linear manifold containing the states reachable from x_0. In spite of this, it is evident the duality of the results are given by Corollaries 1 and 2.

V. Canonical Decomposition

Given a state-affine system and a state x_0, Corollaries 1 and 2 give two linear manifolds both containing x_0. By putting $z = x - x_0$, that is examining the system (5), (31), these linear manifolds began sub-spaces, so that it is possible to decompose the state space into the direct sum of four sub-spaces, as proposed by Kalman (7). In correspondence the system (1), (2) is decomposed into four sub-systems having state spaces which are linear manifolds through x_0.

It is, therefore, possible to extend the well known results on the canonical decomposition of linear systems to the state-affine case.

Hence, for the system (1), (2) and the initial state x_0, consider the corresponding system (5), (31), the span of the states reachable from the origin $P_z(0)$, and the set of the states indistinguishable from the origin $Q_z(0)$. Define the following sub-spaces:

$$\begin{aligned} A &= P_z(0) \cap Q_z(0) \\ P_z(0) &= A \oplus B \\ Q_z(0) &= A \oplus C \\ X &= A \oplus B \oplus C \oplus D \end{aligned} \tag{34}$$

and carry on a change of basis, i.e.

$$\bar{z}(t) = Tz(t) \tag{35}$$

so that the new basis in X is the union of four bases in the sub-spaces A, B, C, D; in this new basis (5) and (31) began:

$$\begin{aligned} \dot{\bar{z}}(t) &= |TF(u(t))T^{-1}|\, \bar{z}(t) + T\,|F(u(t))x_0 + q(u(t))| \\ &\triangleq \bar{F}(u(t))\bar{z}(t) + \bar{F}(u(t))\bar{x}_0 + \bar{g}(u(t)) \end{aligned} \tag{36}$$

where $\bar{x}_0 = Tx_0$ and

$$y(t) = CT^{-1}\bar{z}(t) + Cx_0 \triangleq \bar{C}\bar{z}(t) + Cx_0. \tag{37}$$

In the new basis, $\bar{F}_i$, $\bar{F}_i \cdot \bar{x}_0 + \bar{g}_i$ and consequently the matrices of the system (36), (37) become

$$\bar{F}\,|u(t)| = \begin{pmatrix} \bar{F}_{aa} & \bar{F}_{ab} & \bar{F}_{ac} & \bar{F}_{ad} \\ 0 & \bar{F}_{bb} & 0 & \bar{F}_{bd} \\ 0 & 0 & \bar{F}_{cc} & \bar{F}_{cd} \\ 0 & 0 & 0 & \bar{F}_{dd} \end{pmatrix} \tag{38}$$

$$\tilde{F}\,|u(t)|\,x_0+g\,|u(t)|=\begin{pmatrix}(Fx_0+g)a\\(Fx_0+g)b\\0\\0\end{pmatrix}$$

$$\tilde{C}=(0\quad \tilde{C}_b\quad 0\quad \tilde{C}_d)$$

as it is easy to verify by (12) and (32) with well known algebraic techniques (**1**).

By operating in (1), (2) a change of basis with the same matrix T as in (35), it is easy to verify that the new F, g, C, x_0 obtained are similar to those found in (36), (37) and assume the form (38). Hence the following theorem holds:

Theorem III

Assuming as a basic in the state space, of the system (1), (2), the union of bases on the four sub-spaces A, B, C, D, the functions of the system assume the canonical form (38).

Remark 4. It is easy to verify that

$$P(x_0)=\{x_0+z,\ z\in A\oplus B\} \tag{39}$$

$$Q(x_0)=\{x_0+z,\ z\in A\oplus C\}, \tag{40}$$

hence, in the basis of Theorem III, we have

$$x\in P(\tilde{x}_0)\leftrightarrow \exists x_a,\ x_b: x=\tilde{x}_0+\begin{pmatrix}x_a\\x_b\\0\\0\end{pmatrix} \tag{41}$$

$$x\in Q(\tilde{x}_0)\leftrightarrow \exists x_a,\ x_c: x=\tilde{x}_0+\begin{pmatrix}x_a\\0\\x_c\\0\end{pmatrix} \tag{42}$$

Remark 5. The output of the system (1), (2) from the initial state x_0 coincides with that of system (5), (31) from $z=0$. Therefore, if $\Phi_{u,bb}(t, t_0)$ is the transition matrix associated with the equation

$$\dot{w}(t)=\tilde{F}_{bb}\,|u(t)|\,x(t) \tag{43}$$

the output is expressed by

$$y(t)=\tilde{C}_b\int_0^t \Phi_{u,bb}(t,\tau)\,|\tilde{F}(u(t))\tilde{x}_0+\tilde{g}(u(t))|_b\,\mathrm{d}\tau \tag{44}$$

and, if $\tilde{x}_0$ is partitioned according to (34), we have

$$y(t)=\tilde{C}_b\int_0^t \Phi_{u,bb}(t,\tau)\,|\tilde{F}_{bb}(u(t))\tilde{x}_{0b}+\tilde{F}_{bd}(u(t))\tilde{x}_{0d}$$

$$+\tilde{g}_b(u(t))|\,\mathrm{d}\tau+\tilde{C}_b\tilde{x}_{0b}+\tilde{C}_d\tilde{x}_{0d}. \tag{45}$$

Hence, the input–output map from the initial state x_0 of the system (1), (2) can be expressed in two alternate ways:

(i) By a state-affine system with order equal to $\dim B + \dim D$ as follows:

$$\dot{x}(t) = \begin{pmatrix} \tilde{F}_{bb}(u(t)) & \tilde{F}_{bd}(u(t)) \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} \tilde{g}_b(u(t)) \\ 0 \end{pmatrix}$$

$$y(t) = (\tilde{C}_b \; \tilde{C}_d)x(t) \tag{46}$$

with

$$x(0) = \begin{pmatrix} \tilde{x}_{0b} \\ \tilde{x}_{0d} \end{pmatrix}.$$

(ii) By a system that is not a variable structure one with order equal to dim B as follows:

$$\begin{aligned} \dot{x}(t) &= \tilde{F}_{bb}(u(t))x(t) + |\tilde{F}_{bd}(u(t))\tilde{x}_{0d} + \tilde{q}_b(u(t))| \\ y(t) &= \tilde{C}_b x(t) + \tilde{C}_d \tilde{x}_{0d} \end{aligned} \tag{47}$$

with $x(0) = \tilde{x}_{0b}$.

In both cases, the input–output map depends not only on the b sub-system but also, in some manner, on the d part of the decomposition.

VI. Example

The following state-affine system is given: i.e.

$$\begin{aligned} \dot{x}(t) &= \frac{1}{2}\begin{pmatrix} 2u - u^2 & u^2 \\ -u^2 & 2u + u^2 \end{pmatrix} x(t) + \frac{1}{2}\begin{pmatrix} u + u^2 \\ -u + u^2 \end{pmatrix} \\ y(t) &= (1 \quad 1)x(t) \end{aligned} \tag{48}$$

with

$$x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $u_1 = 1$ and $u_2 = 2$ input values that satisfy the first of (11) and (24); hence, it is easy to verify that

$$F_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \quad F_2 = \frac{1}{2}\begin{pmatrix} 0 & 4 \\ -4 & 8 \end{pmatrix},$$

$$|Fx_0 + g|_1 = \frac{1}{2}\begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad |Fx_0 + g|_2 = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

and

$$\mathrm{gen}_{F_1,F_2}(|Fx_0 + g|_1 \; |Fx_0 + g|_2) = \begin{pmatrix} \frac{3}{2} & 5 & 3 & 12 & 5 & 20 \\ \frac{3}{2} & 5 & 3 & 12 & 5 & 20 \end{pmatrix}.$$

By a change of basis as in (35), with

$$T=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

it follows that

$$F\,|u(t)|=\begin{pmatrix} u & u^2 \\ 0 & u \end{pmatrix}, \quad F\,|u(t)|\,x_0+g\,|u(t)|=\begin{pmatrix} 2u^2+u \\ 0 \end{pmatrix}$$

and $C=(1 \quad 0)$.

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