PERGAMON

Automatica 37 (2001) 1585–1592

automatica

www.elsevier.com/locate/automatica

Brief Paper

The weighted incremental norm approach: from linear to nonlinear H_∞ control[☆]

V. Fromion[a,*], S. Monaco[b], D. Normand-Cyrot[c]

[a]INRA–Laboratoire d'Analyse des Systèmes et de Biométrie, 2 place P. Viala, 34060 Montpellier, France
[b]Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", 18 Via Eudossiana, 00184 Roma, Italy
[c]Laboratoire des Signaux et Systèmes, Ecole Supérieure d'Eléctricité, Plateau de Moulon, 91190 Gif-sur-Yvette, France

Received 17 December 1997; revised 19 September 2000; received in final form 19 January 2001

Abstract

Weighted induced norms can be used to handle robustness and sensitivity requirements in the linear context as shown by Zames in its seminal work. This paper deals with an attempt to extend linear H_∞ control concepts to the nonlinear context making use of a weighted incremental norm. It is shown how these concepts make it possible to handle basic requirements such as robust stability, disturbance attenuation and steady state behaviors. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Nonlinear systems analysis; Robust control; Sensitivity function; Nonlinear H_∞ control; Gain-scheduling

1. Introduction

H_∞ control is issued from the effort of formalizing in mathematical terms performance and robustness requirements. In the linear context, a significant part of the activities is devoted to reformulate and generalize the classical control concepts introduced by Black, Bode, and Horowitz such as phase and gain margins and the sensitivity concept. Zames (1981) shows how weighted induced norms can be used to handle both robustness and sensitivity requirements. Is it possible to extend this idea to the nonlinear context? That is the question studied hereafter.

The nonlinear extension of the H_∞ optimization problem is investigated in the recent literature in the $\mathscr{L}_2$ framework through polynomial expansions (Foias & Tannenbaum, 1989), dissipativity techniques/nonlinear differential game arguments (Basar & Bernhard, 1991), linear H_∞ methods applied to systems perturbed by nonlinear uncertainties (Becker, Packard, Philbrick, & Balas, 1993). A robustness theory for nonlinear systems with general unstructured uncertainties (graph perturbation, gap metric) was presented in the framework of differentiable/incremental norms by Georgiou (1993).

In this paper, providing a unified view of results stated by the authors (see Fromion, 1995, 1997; Fromion, Monaco, & Normand-Cyrot, 1995, 1996; Fromion, Scorletti, & Ferreres, 1999), we show that a possible way to extend the H_∞ approach to the nonlinear context can be the weighted incremental norms. For this purpose, we investigate some of the aspects considered by Zames (1981). The result is that the incremental framework allows us to take into account not only the classical linear requirements, such as robust stability with respect to unstructured uncertainties, attenuation with respect to output perturbations and sensitivity, but also specific problems associated with the nonlinear nature of the plant such as initial condition uncertainties and steady state properties with respect to specific classes of inputs (constant or periodic).

The paper is organized as follows. The robustness problem is addressed in Section 2 where robustness against unstructured uncertainties is restated in terms of an incremental test. In Section 3, it is shown how the attenuation problem can be reduced to the minimization of a weighted incremental norm. In Section 4, weighted induced norms, namely $\mathscr{L}_2$ or incremental gains, are

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor F. Jabbari under the direction of Editor Roberto Tempo. Supported by the Italian Space Agency, Study Project ASI RS 9417.

* Corresponding author. Tel.: + 33-4-99-61-22-38; fax: + 33-4-67-52-14-27.

E-mail address: fromion@ensam.inra.fr (V. Fromion).

0005-1098/01/$ - see front matter
PII: S0005-1098(01)00106-6

used to handle, in a weak sense, constraints on the behavior of the closed-loop system. We point out some limitations of the $\mathscr{L}_2$ gain approach and then show how, under suitable assumptions, incremental boundedness allows to bypass these limitations. Section 5 studies the connection of the proposed approach with linear time-varying H_∞ control. The classical gain scheduling (Shamma, 1988; Rugh, 1991) is also re-interpreted as an approximation of an incremental objective.

Notations and definitions. The notations and terminology, here used, are classical in the input–output context (see Willems, 1971; Desoer & Vidyasagar, 1975). The $\mathscr{L}_2$-norm of $f:[t_0,\infty)\mapsto\mathbb{R}^n$ is $\|f\|_2=\sqrt{\int_{t_0}^{\infty}\|f(t)\|^2\,dt}$. The *causal truncation* at $T\in[t_0,\infty)$, denoted by P_Tf gives $P_Tf(t)=f(t)$ for $t\leqslant T$ and 0 otherwise. The *extended space*, $\mathscr{L}_2^e$ is composed with the functions whose causal truncations belong to $\mathscr{L}_2$. For convenience, $\|P_Tu\|_2$ is denoted by $\|u\|_{2,T}$.

In the sequel, we consider systems exhibiting the differential representation:

$$\Sigma\begin{cases}\dot{x}(t)=f(x(t),u(t)),\\ y(t)=h(x(t),u(t)),\\ x(t_0)=x_0,\end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$, $y(t)\in\mathbb{R}^m$, and $u(t)\in\mathbb{R}^p$. f and h, defined from $\mathbb{R}^n\times\mathbb{R}^p$ into $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, are assumed to be C^1 and uniformly Lipschitz. The unique solution $x(t)=\phi(t,t_0,x_0,u)$ belongs to $\mathscr{L}_2^e$ for all $x_0\in\mathbb{R}^n$ and for any $u\in\mathscr{L}_2^e$. It is assumed that there exists x_{0e} such that $f(x_{0e},0)=0$ and $h(x_{0e},0)=0$, i.e. the system initialized at x_{0e} is unbiased, $\Sigma(0)=0$.

Σ is said to be a weakly finite gain stable system if there exist $\gamma\geqslant 0$, $\beta\geqslant 0$ such that $\|\Sigma(u)\|_2\leqslant\gamma\|u\|_2+\beta$ for all $u\in\mathscr{L}_2$. Σ is said to be finite gain stable when $\beta=0$. The gain of Σ coincides with the minimum value of γ and is denoted by $\|\Sigma\|_i$. Σ has a finite incremental gain if there exists $\eta\geqslant 0$ such that $\|\Sigma(u_1)-\Sigma(u_2)\|_2\leqslant\eta\|u_1-u_2\|_2$ for all $u_1,u_2\in\mathscr{L}_2$. The incremental gain of Σ coincides with the minimum value of η and is denoted by $\|\Sigma\|_\Delta$. Σ is said to be incrementally stable if it is stable, i.e. it maps $\mathscr{L}_2$ to $\mathscr{L}_2$, and has a finite incremental gain.

We consider in the sequel, the nonlinear feedback system depicted in Fig. 1, where G,K,F are nonlinear causal operators from $\mathscr{L}_2^e$ into $\mathscr{L}_2^e$, representing respectively, the plant, the compensator and the feedback, and where r,e,u and y, which belong to $\mathscr{L}_2^e$, denote respectively, the system input, the error signal, the plant input, and the system output. The closed-loop system is assumed to be well-posed and the input–output map between the system input and the system output is denoted by H_{yr} and is given by $GK(I+FGK)^{-1}$.

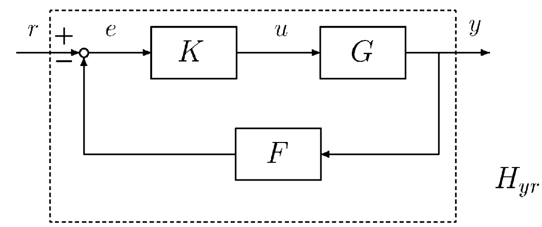

Fig. 1. The nonlinear feedback system.

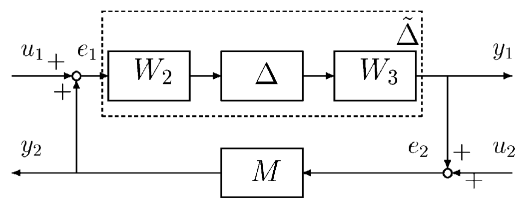

Fig. 2. A perturbed closed-loop system.

2. Robust stability against unstructured uncertainties

The description of unstructured uncertainties through the use of suitable weights is classical in the linear context (Safonov, 1980; Zames, 1981; Doyle, Wall, & Stein, 1982). This is for example the usual way to take into account uncertainties due to actuator dynamics, output sensor errors, high-frequency neglected dynamics (bending modes) or some limitations of the system such as gain margin requirements through the use of an input multiplicative error. Such a description, not depending on the nature of the nominal plant, can be assumed to hold in the nonlinear context too.

With this in mind, we will consider additive uncertainties, i.e. $\tilde{G}=G+\tilde{\Delta}$, multiplicative uncertainties, i.e. $\tilde{G}=G(I+\tilde{\Delta})$, or some other types as in (Doyle et al., 1982) and we will assume that $\tilde{\Delta}$ belongs to Ω_Δ defined by

$$\Omega_\Delta\equiv\{\tilde{\Delta}=W_3\Delta W_2\,|\,\|\Delta\|_\Delta<1\}, \tag{2}$$

where W_2 and W_3 are two causal and incrementally stable operators.

We can now formulate the robustness problem as the property of the induced norm of the system augmented with the weighting functions W_2 and W_3. The assumption set on uncertainty allows us to represent the perturbed system as depicted in Fig. 2, where M is the generic nominal closed-loop system. The following theorem represents a first extension of a known linear result.

Theorem 1 (Fromion, Monaco, & Normand-Cyrot, 2000). *If M is incrementally stable and if $\|W_2MW_3\|_\Delta\leqslant 1$ then the closed-loop system of Fig. 2 is incrementally stable for any $\tilde{\Delta}$ belonging to Ω_Δ.*

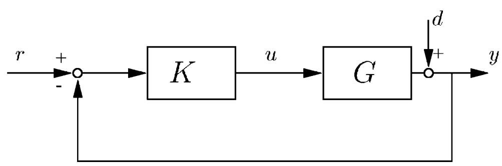

Fig. 3. The perturbed closed-loop system.

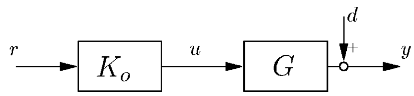

Fig. 4. The perturbed equivalent open-loop system.

3. Disturbance attenuation problem

The use of feedback control schemes is mainly linked to their ability to reduce the effect of non measurable perturbations or to shrink model uncertainties: the desensitivity property (see e.g. Desoer & Wang, 1980; Zames, 1981). Two major types of desensitivities are classically considered: the infinitesimal desensitivity, related to small perturbations, and the comparison desensitivity when no restrictions concerning the size of perturbations are present. As it is pointed out by Desoer and Wang (1980), on the basis of Taylor type expansion arguments, it is possible to link one to the other.

In the following, it is shown that the desensitivity requirement can be reformulated as the minimization of the incremental norm of a suitable weighted map. The reader is referred to Desoer and Wang (1980) for a complete presentation of the desensitivity problem in a nonlinear context. In the sequel, we will just consider the output disturbance problem (the other cases presented by Desoer and Wang (1980) can be worked out as well).

As Desoer and Wang (1980), we assume without loss of generality that the feedback map is the identity, i.e. $F = I$, and we associate with the closed-loop system depicted in Fig. 3 an "equivalent" open-loop map, $H_{o_{yr}}$ depicted in Fig. 4. If the open-loop controller is given by $K_o = K(I + GK)^{-1}$, then the open-loop system in Fig. 4, which maps the inputs (r, d) in $\mathscr{L}_2^e \times \mathscr{L}_2^e$ to the output y in $\mathscr{L}_2^e$, satisfies for all $r \in \mathscr{L}_2^e$ and for $d = 0$, the following equality:

$$H_{o_{yr}}(r, 0) \triangleq H_{yr}(r, 0),$$

where H_{yr} is the system in Fig. 3 which maps inputs (r, d) which belong to $\mathscr{L}_2^e \times \mathscr{L}_2^e$ to the output y which also belongs to $\mathscr{L}_2^e$.

We now compute the effect induced by the output perturbations on the open-loop system:

$$\delta H_{o_{yr}}(r, d) = GK(I + GK)^{-1}(r) + d - GK(I + GK)^{-1}(r) = d \tag{3}$$

whereas the closed-loop configuration gives

$$\delta H_{yr}(r, d) = GK(I + GK)^{-1}(r - d) + d - GK(I + GK)^{-1}(r) \tag{4}$$

$$= (I + GK)^{-1}(r) - (I + GK)^{-1}(r - d) \tag{5}$$

since $GK(I + GK)^{-1} = I - (I + GK)^{-1}$.

The main interest of the feedback control strategy stands in its capability of reducing the effect of disturbances. In mathematical terms, the feedback has a desensisitivity effect if the following inequality is satisfied:

$$\|\delta H_{yr}(r, d)\|_{2,T} < \|\delta H_{o_{yr}}(r, d)\|_{2,T}.$$

Unfortunately, for realistic systems, this inequality cannot be satisfied for any inputs and disturbances belonging to $\mathscr{L}_2^e$. Indeed, like in the linear context (Zames, 1981), one has the following theorem:

Theorem 2 (Fromion et al., 2000). *Consider the closed-loop system in Fig.* 1 *with $F = I$. If the open-loop operator GK is strictly causal then $\|(I + GK)^{-1}\|_\Delta \geqslant 1$.*

This theorem implies that there exist $r, d \in \mathscr{L}_2^e$ such that $\|S(r - d) - S(r)\|_{2,T} \geqslant \|d\|_{2,T}$: where $S = (I + GK)^{-1}$, i.e. a disturbance with respect to which the feedback law does not behave better than the open loop strategy, $\|\delta H_{yr}(r, d)\|_{2,T} \geqslant \|\delta H_{o_{yr}}(r, d)\|_{2,T}$.

Following this preliminary remark, the interest of feedback law is necessarily limited to a specific class of perturbations, say $P^e \subset \mathscr{L}_2^e$. The use of the feedback control law could be justified if (and only if) there exists $\varepsilon \ll 1$ such that

$$\|\delta H_{yr}(r, d)\|_{2,T} \leqslant \varepsilon \|\delta H_{o_{yr}}(r, d)\|_{2,T}, \tag{6}$$

for any $d \in P^e \subset \mathscr{L}_2^e$ and any $r \in \mathscr{L}_2^e$.

We now show how this requirement can be formulated in terms of a weighted incremental criteria. To this purpose, as in the H_∞ approach (see Zames, 1981), we assume that the set of possible disturbances where desensitivity must be achieved can be defined by

$$P^e = \{d \in \mathscr{L}_2^e \mid \|W_p^{-1}(d) - W_p^{-1}(r + d)\| \leqslant \varepsilon \|d\| \text{ for any } r \in \mathscr{L}_2^e\},$$

where W_p and W_p^{-1} are two causal and incrementally stable operators.

Theorem 3 (Fromion et al., 2000). *Consider the nonlinear feedback system depicted in Fig.* 3. *If $\|(I + GK)^{-1}W_p\|_\Delta \leqslant 1$ then $\|\delta H_{yr}(r, d)\|_{2,T} \leqslant \varepsilon \|\delta H_{o_{yr}}(r, d)\|_{2,T}$ for any $d \in P^e \subset \mathscr{L}_2^e$ and any $r \in \mathscr{L}_2^e$.*

Note that the use of nonlinear weighting functions for specifying attenuation requirements allows to take into account operating points defined by the values of r and d

(a remark in Section 5, after Proposition 11, clarifies this aspect).

The ability of the feedback to reject the effect of uncertainties can be studied in the same way. As a matter of fact, the uncertainties generate signal perturbations which should be rejected as much as possible. Since the uncertainties modify the input–output properties, the rejection is obtained only if the perturbed maps possesses suitable properties.

4. Input–output performance

Input–output performance is related to suitable properties of the input–output maps together with requirements on the outputs corresponding to prefixed input signals. In this context, a classical requirement concerns the steady-state behaviors associated to constant or periodic references together with their maintenance under perturbations. Such a property directly follows from the internal stability when dealing with linear systems: as a matter of fact the stability of the null trajectory ensures the asymptotic rejection of any perturbation acting in finite time or vanishing at the infinity. This is no longer the case in a nonlinear context where external and internal stability on any output trajectory does not follow from the stability with respect to a particular one.

4.1. Black formulae

Desoer and Wang (1980) describe the performance as the ability of a closed-loop system to asymptotically minimize the gain between references and error signals. This approach is recalled in the sequel. Denoting by $R_d^e \subset \mathscr{L}_2^e$ the set of inputs of interest (e.g. sinusoids, steps, ramps, ...), one sets:

Definition 4. Asymptotic performance of the system depicted in Fig. 1 is satisfied on R_d^e if for all $r \in R_d^e$, there exists $T_0 \geqslant t_0$ such that for all $T \geqslant T_0$, one has: $\|(I+FGK)^{-1}r\|_{2,T} \leqslant \|r\|_{2,T}$.

Definition 4 ensures that the relation $FH_{yr} \approx I$ is asymptotically satisfied on R_d^e and indicates that H_{yr}, restricted to the domain of interest, is essentially specified by F and is quite independent of G. This is the nonlinear equivalent of the well-known Black formulae (Desoer & Wang, 1980).

The asymptotic performance can now be specified in terms of a weighted $\mathscr{L}_2$ norm as pointed out below. We assume the existence of an invertible causal and $\mathscr{L}_2$ stable operator W_I and $T_0 \geqslant t_0$ such that, for all $r \in R_d^e$ and all $T \geqslant T_0$, one has

$$\|W_I^{-1}(r)\|_{2,T} \leqslant \|r\|_{2,T}. \tag{7}$$

Theorem 5 (Fromion et al., 2000). *If the weighting function satisfies condition* (7) *and if* $\|(I+FGK)^{-1}W_I\|_i \leqslant 1$ *then, the closed-loop system in Fig.* 1 *has the asymptotic performance property on* R_d^e.

The minimization of an $\mathscr{L}_2$ gain is not enough. A first aspect which limits the validity of this approach is the unbiasedness assumption. A nominal system can always be assumed to be unbiased setting $\tilde{H}(u) = H(u) - H(0)$ but this unbiased assumption cannot be maintained when the initial condition is modified. Indeed, as pointed out by Hill and Moylan (1980), a change in the initial condition makes the system weakly $\mathscr{L}_2$-gain stable. In this case, this implies that there exists a suitable $\beta \geqslant 0$ such that $\|(I+FGK)^{-1}r\|_{2,T} \leqslant \|W_I^{-1}(r)\|_{2,T} + \beta$ and thus the value of β limits the performance of the system. This means that the validity of Theorem 5 for the nominal system does not guarantee the robustness with respect to the initial condition of the asymptotic performance. This first limitation is overcome when an incremental type criteria is used. As a matter of fact, under the weak assumption that the "perturbed" initial condition is reachable from the initial one, i.e. there exists an input u which allows to reach the perturbed initial condition from the nominal one under a finite time, it is possible to claim (see Fromion et al., 1996, Lemma 1) that the nonlinear operator associated to the perturbed initial condition satisfies the same weighted criterion and thus it has the asymptotic performance property on R_d^e.

4.2. Steady state properties

The existence of a unique constant steady state behavior associated to any constant input together with the Lyapunov stability of the unperturbed trajectory represents in many cases a minimal requirement. In this context, it is possible to prove that $\mathscr{L}_2$ gain type criteria do not allow to guarantee such properties other than for the null input (see the counter-example given by Fromion et al., 1999).

From (Fromion, 1997) incrementally bounded systems possess a unique steady-state behavior if the state space realization of Σ possesses some suitable observability properties:

Theorem 6 (Fromion, 1997). *Let* Σ *be a dynamical system with a finite incremental gain. If the unperturbed motion, associated with* $x_{0r} \in \mathbb{R}^n$ *and* $u_r \in \mathscr{L}_2^e$, *is uniformly observable, then for any* $\tilde{u}_r \in \mathscr{L}_2^e$ *such that* $u_r - \tilde{u}_r$ *belongs to* $\mathscr{L}_2$, *one has* $\lim_{t\to\infty} \|\phi(t,t_0,x_{0r},u_r) - \phi(t,t_0,x_{0r},\tilde{u}_r)\| = 0$.

We note that contrarily to $\mathscr{L}_2$-gain stable systems, incrementally bounded systems possess suitable properties for a large class of inputs since the previous result holds true for any input in $\mathscr{L}_2^e$ and thus, as an example, for the constant inputs.

A second interesting result which is pointed out concerns the state evolution under specific inputs.

Theorem 7 (Fromion, 1997). *Let Σ be a stationary dynamical system with a finite incremental gain. Assume that the unperturbed motion of Σ, associated with $x_{0r} \in \mathbb{R}^n$ and $u_r \in \mathscr{L}_2^e$, is uniformly observable and its state space is uniformly isotropically reachable from x_{0r}. Then, if u_r is a periodic input, the unperturbed motion is asymptotically periodic. Moreover, there exists at least one initial condition such that the motion and the output associated with this input are periodic functions.*

Theorem 7 implies, under some assumptions on the state space realization of the system, that it is possible to associate an equilibrium point with each constant input.

We conclude this section by pointing out that the incremental criterium under some assumptions concerning the state space realization of the closed-loop system, ensures Lyapunov stability of unperturbed motions. This property provides a better characterization of the robustness with respect to the initial state or for specifying the effect of past inputs over the future system behaviors.

Theorem 8 (Fromion, 1997). *Let Σ be a dynamical system with a finite incremental gain. If the unperturbed motion of Σ, associated with $x_{0r} \in \mathbb{R}^n$ and $u_r \in \mathscr{L}_2^e$, is uniformly observable and if the state space of Σ is uniformly isotropically reachable from x_{0r} then this unperturbed motion is uniformly globally asymptotically stable.*

5. Connections with non-stationary and stationary H_∞ control

We illustrate, in this section, the connection between requirements set in terms of weighted incremental norms and some local ones associated with the linearizations of the operator. For this purpose, we first recall an important result in the context of incrementally bounded systems, which links the incremental gain of a nonlinear operator to the norm of its derivatives. We show that satisfying a weighted incremental type criterium is equivalent to satisfy an infinity of non-stationary suboptimal H_∞ criteria. Then, the type of stationary and of non-stationary H_∞ criteria associated with the linear approximation is discussed.

For the sake of clarity, we first recall known results about the differentiability of nonlinear operators defined over functional spaces.

Definition 9. Given a causal operator Σ, defined from $\mathscr{L}_2^e$ into $\mathscr{L}_2^e$, let $u_0 \in \mathscr{L}_2^e$ and assume the existence for any $T \in [t_0, \infty)$ and any $h \in \mathscr{L}_2^e$ of a continuous linear operator $D\Sigma_G[u_0]$ from $\mathscr{L}_2^e$ into $\mathscr{L}_2^e$ such that

$$\lim_{\lambda \downarrow 0} \left\| \frac{\Sigma(u_0 + \lambda h) - \Sigma(u_0)}{\lambda} - D\Sigma_G[u_0](h) \right\|_{2,T} = 0,$$

then $D\Sigma_G[u_0]$ is called the Gâteaux derivative (the linearization) of Σ at u_0.

When the system is generated by differential equations, Definition 9 corresponds to the usual linearization concept. Under the assumption made on f and h in Eq. (1), i.e. uniformly Lipschitz and C^1, $y = \Sigma(u)$ has a Gâteaux derivative[1] for all $u \in \mathscr{L}_2^e$. Moreover, its linearization along the input $u_r(t)$, denoted by $\bar{y} = D\Sigma_G[u_r](\bar{u})$, satisfies the differential equations

$$\begin{cases} \dot{\bar{x}}(t) = \dfrac{\partial f}{\partial x}(x_r(t), u_r(t))\bar{x}(t) + \dfrac{\partial f}{\partial u}(x_r(t), u_r(t))\bar{u}(t), \\ \bar{y}(t) = \dfrac{\partial h}{\partial x}(x_r(t), u_r(t))\bar{x}(t) + \dfrac{\partial h}{\partial u}(x_r(t), u_r(t))\bar{u}(t), \\ \bar{x}(t_0) = 0, \end{cases} \tag{8}$$

where $x_r(t)$ is the solution of (1) under input $u_r(t)$.

The theorem recalled below is a key result in the context of nonlinear control. It sets a strong connection between the incremental norm and the local properties associated with the derivative of a nonlinear system.

Theorem 10 (Willems, 1971).[2] *Let us assume that a causal operator Σ defined from $\mathscr{L}_2^e$ into $\mathscr{L}_2^e$ has a Gâteaux derivative at each point u_0 of $\mathscr{L}_2^e$. Σ has a finite incremental gain if and only if there exists a finite constant η such that for any $u_0 \in \mathscr{L}_2^e$ and any $T \geqslant t_0$, one has*

$$\|P_T D\Sigma_G[u_0]\|_i \leqslant \eta.$$

Moreover $\|\Sigma(u_1) - \Sigma(u_2)\|_\Delta = \sup_{u_0} \|P_T D\Sigma_G[u_0]\|_i$.

With this in mind we can now point out the connections between the weighted incremental approach and H_∞ control. Let $M_{zw} = W_o H W_i$ be the augmented plant where W_i and W_o are the input and output weighting functions associated with robustness and performance requirements. We assume that the augmented system is described by a differential equation with C^1 and globally Lipschitz drift and output functions (this ensures the existence of the Gâteaux derivative of the augmented system). From Theorem 10, one deduces:

Proposition 11. *If the augmented system, $M_{zw} = W_o H W_i$, possesses a Gâteaux derivative for every input in $\mathscr{L}_2^e$ then*

[1] It is possible to prove that if f and g are not linear functions of their arguments that the system is not Fréchet differentiable on $\mathscr{L}_2^e$.

[2] The proof provided by Willems (1971), even if Σ is Gâteaux differentiable and not Fréchet differentiable, can be easily extended to our case.

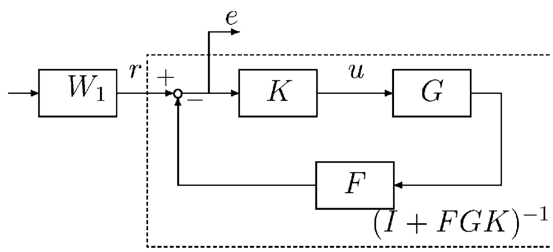

Fig. 5. The augmented plant.

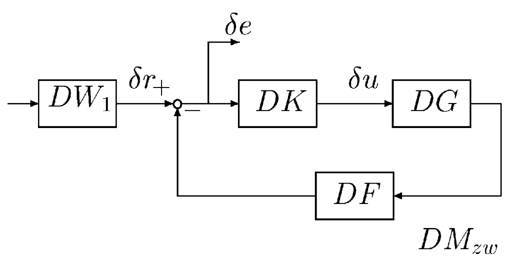

Fig. 6. Linearization of the augmented plant.

$\|M_{zw}\|_{\Delta} \leqslant 1$, *if and only if*

$$\|DW_{\mathrm{oG}}[H(W_{\mathrm{i}}(w_0))]DH_{\mathrm{G}}[W_{\mathrm{i}}(w_0)]DW_{\mathrm{iG}}[w_0]\|_i \leqslant 1,$$
$$\forall w_0 \in \mathscr{L}_2^{\mathrm{e}}. \tag{9}$$

Recalling that $DM_{zw\mathrm{G}}[w_0]$ is a linear time-varying operator, Proposition 11 shows that solving a weighted incremental problem is equivalent to solving an infinite number of linear time-varying weighted induced norm problems. It is worth noting that the constraints (9) are satisfied if (and only if) an infinite number of linear time-varying weighted H_∞ constraints are satisfied.

The non-stationary characteristics of the induced norm criterium of Proposition 11 is discussed through a simple example described by Fig. 5. Given a small variation $\delta r(t)$ of the system input $r(t)$, the tracking error variation can then be approximated on a finite time interval

$$e = S(r + \lambda\delta r) - S(r) \approx DS_{\mathrm{G}}[r](\lambda\delta r),$$

where $S = (I + FGK)^{-1}$. Let us here assume that the performance requirements are taken into account by using a linear weighting input operator, $W_{\mathrm{i}} = W_I$. This weighting function, assumed to be causal and invertible, satisfies the following relation (see Section 4) $\|P_T W_I^{-1}(r)\|_2 \leqslant \|P_T r\|_2$. If the condition of Proposition 11 is satisfied, it can be claimed that $\|DS_{\mathrm{G}}[W_I(w_0)]W_I\|_i \leqslant 1$.

The above relation represents an H_∞ time-varying constraint (see Fig. 6), which ensures for any $\delta r \in R_d^{\mathrm{e}}$ that there exists a time, $T_0 \geqslant t_0$, such that for all $T \geqslant T_0$, one has $\|P_T \delta e\|_2 \leqslant \|P_T \delta r\|_2$. Note that

$$\|P_T DS_{\mathrm{G}}[W_I(w_0)](\delta r)\|_2 \leqslant \|P_T \; W_I^{-1}(\delta r)\|_2 \leqslant \|P_T \delta r\|_2.$$

In a performance context, Proposition 11 can be interpreted in two different ways:

- as a constraint on the linearizations of the system along the trajectory defined by $W_I(w_0)$. Consequently, this guarantees a good behavior of the nonlinear system along this trajectory despite small perturbations belonging to R_d^{e}.
- as a constraint on the output variations with respect to small input variations. For example, the output associated with a step input can be interpreted as the succession of responses to small step-inputs associated with each linearization of the nonlinear system along the trajectory generated by this step. The quality of this output is directly linked to the linearizations of the weighting functions W_I.

In the approach proposed by Shamma (1988), one has to check whether the gain scheduling system satisfies a criterion of the same type as in Eq. (9). Our approach from a different point of view highlights the interest of the study of linear parameter varying (LPV) plants in a nonlinear context.

In the rest of this paragraph, we will show a close connection between the incremental approach and the classical gain-scheduling technique. For this purpose, we restrict our attention to a specific class of linearizations, namely the time invariant ones. We then define Z_{e}, the set of equilibrium points associated with any constant input:

$$Z_{\mathrm{e}} = \{(x_{\mathrm{e}}, u_{\mathrm{e}}) \in \mathbb{R}^n \times \mathbb{R}^p \mid \phi(t, t_0, x_{\mathrm{e}}, u_{\mathrm{e}}) = x_{\mathrm{e}}, \ \forall t \geqslant t_0\},$$

where ϕ is the state transition map of Σ.

Theorem 12 (Fromion et al., 1996). *Let Σ be the system given by* (1) *with a finite incremental gain η. Let u_{e} be a constant input and x_{e} its associated equilibrium point. If x_{e} is reachable from x_0 then the linearization of Σ, given by the following linear time invariant system*

$$D\Sigma_{\mathrm{G}}(u_{\mathrm{e}}) \begin{cases} \dot{\bar{x}}(t) = F\bar{x}(t) + G\bar{u}(t), \\ \bar{y}(t) = H\bar{x}(t) + J\bar{u}(t), \\ \bar{x}(t_0) = 0. \end{cases}$$

$F = \partial f/\partial x(x_{\mathrm{e}}, u_{\mathrm{e}})$, $G = \partial f/\partial u(x_{\mathrm{e}}, u_{\mathrm{e}})$, $H = \partial h/\partial x(x_{\mathrm{e}}, u_{\mathrm{e}})$, $J = \partial h/\partial u(x_{\mathrm{e}}, u_{\mathrm{e}})$, has a finite $\mathscr{L}_2$ gain less than or equal to η, i.e. $\|D\Sigma_{\mathrm{G}}[u_{\mathrm{e}}]\|_i \leqslant \eta$.

This result sets a direct connection between our nonlinear framework and the classical gain scheduling techniques, especially with the approaches based on the extended linearization (Rugh, 1991), where some properties are imposed to the linear time-invariant linearizations of the system associated with constant inputs. The result of Theorem 12 renews the interest of incremental norm versus $\mathscr{L}_2$ since, with reference to the counter-example presented in Fromion, Scorletti, and Ferreres (1999) we point out that $\mathscr{L}_2$-gain stability does not necessarily guarantee the stability of the linearizations associated with constant inputs. Furthermore, with respect to the weighted incremental norm approach and with reference to the augmented system previously defined, which has norm less than 1, i.e. $\|M_{zw}\|_\Delta \leqslant 1$), Theorem 12 ensures that all the linearizations satisfy an H_∞ criterion. This criterion is specified at each equilibrium point by the stationary linearization of the nonlinear weighting functions, i.e.

$$\|DW_{\mathrm{oG}}[H(W_{\mathrm{i}}(w_0))]DH_{\mathrm{G}}[W_{\mathrm{iG}}(w_0)]DW_{\mathrm{iG}}[w_0]\|_i \leqslant 1,$$

where $DW_{\mathrm{oG}}[H(W_{\mathrm{i}}(w_0))]$, $DH_{\mathrm{G}}[W_{\mathrm{i}}(w_0)]$ and, $DW_{\mathrm{iG}}[w_0]$ are linear time invariant systems. This last fact has interesting connections with the work presented by Hyde and Glover (1993).

6. Conclusion

It has been shown how weighted incremental norms can be used to handle, in a nonlinear context, basic requirements such as robust stability, disturbance attenuation and steady state behaviors. The strong connections between requirements set in terms of a weighted incremental norm and local requirements, given with reference to the linear approximations of the plant, are also pointed out.

The practical interest of this approach is illustrated in (Fromion et al., 1999) where the case study of a PI controlled missile is investigated.

References

Basar, T., & Bernhard, P. (1991). *H_∞—optimal control and related minimax design problems: A dynamical game approach*. Boston, MA: Birkhauser.

Becker, G., Packard, A., Philbrick, D., & Balas, G. (1993). Control of parametrically dependent linear systems: A single quadratic Lyapunov approach. *Proceedings of the 1993 American Control Conference*, Vol. 3.

Desoer, C. A., & Vidyasagar, M. (1975). *Feedback systems: Input–output properties*. New York: Academic Press.

Desoer, C., & Wang, Y. (1980). Foundations of feedback theory for nonlinear dynamical systems. *IEEE Transactions on Circuits and Systems*, *27*, 104–123.

Doyle, J., Wall, J. E., & Stein, G. (1982). Performance and robustness analysis for structured uncertainties. *Proceedings of the IEEE Conference on Decision and Control.*

Foias, C., & Tannenbaum, A. (1989). Weighted optimization theory for non linear systems. *SIAM Journal on Control and Optimization*, *27*, 842–860.

Fromion, V. (1995). *Une approche incrementale de la robustesse non linéaire*; *application au domaine de l'aéronautique.* Ph.D. thesis, Université de Paris Sud Orsay.

Fromion, V. (1997). Some results on the behavior of Lipschitz continuous systems. *Proceedings of the European Control Conference.*

Fromion, V., Monaco, S., & Normand-Cyrot, D. (1995). A possible extension of H_∞ control to the nonlinear context. *Proceedings of the IEEE Conference on Decision and Control.*

Fromion, V., Monaco, S., & Normand-Cyrot, D. (1996). Asymptotic properties of incrementally stable systems. *IEEE Transactions on Automatic Control*, *41*, 721–723.

Fromion, V., Monaco, S., & Normand-Cyrot, D. (2000). *The weighted incremental norm approach*: *From linear to nonlinear H_∞ control.* Technical Report, LASB, INRA, Montpellier.

Fromion, V., Scorletti, G., & Ferreres, G. (1999). Nonlinear performance of a PI controlled missile: An explanation. *International Journal of Robust and Nonlinear Control*, *9*(8), 485–518.

Georgiou, T. (1993). Differential stability and robust control of nonlinear systems. *Mathematics of Control, Signal, and Systems*, *6*, 289–306.

Hill, D. J., & Moylan, P. J. (1980). Connections between finite-gain and asymptotic stability. *IEEE Transactions on Automatic Control*, *AC-25*, 931–936.

Hyde, R. A., & Glover, K. (1993). The application of scheduled H_∞ controllers to a VSTOL aircraft. *IEEE Transactions on Automatic Control*, *38*, 1021–1039.

Rugh, W. J. (1991). Analytical framework for gain scheduling. *IEEE Control Systems and Mag.*, *11*(1), 79–84.

Safonov, M. G. (1980). *Stability and robustness of multivariable feedback systems*. Cambridge, MA: MIT Press.

Shamma, J. (1988). *Analysis and design of gain scheduled control systems*. Ph.D. thesis, Department of Mechanical Engineering, M.I.T.

Willems, J. C. (1971). The analysis of feedback systems *Research Monographs*, Vol. 62. Cambridge, MA: MIT Press.

Zames, G. (1981). Feedback and optimal sensitivity: Model reference transformations, mutiplicative seminorms, and approximate inverse. *IEEE Transactions on Automatic Control*, *26*(2), 301–320.

Vincent Fromion born in 1967, received the Engineer degree from Ecole Supérieure d'Electricité (Supelec-France) in 1990 and the Ph.D. degree from the University of Orsay in 1995. Since 1999 he has been with the Institut National de la Recherche Agronomique (INRA) where he is presently a Chargé de Recherche. During the academic year 1996–1997 he was a Post-doc visiting with the Dipartimento di Informatica e Sistemistica, University of Rome La Sapienza and with the Department of Electrical Engineering, University of Southern California, Los Angeles.

Salvatore Monaco born in 1951, received the "Laurea" degree in Electrical Engineering in 1974. Research fellow from 1976 to 1979, Associate Professor from 1979 to 1986, Full Professor, University of L'Aquila until 1989 where he was member of the board of directors. From 1989 he is Full Professor at the University of Rome La Sapienza where teaches System Theory and Automatic Control. He is author and co-author of nearly 120 scientific publications in the area of Control and Systems Theory; one of them has been nominated as Outstanding Paper in the IEEE Transactions on Automatic Control for the years 1981–1982. From 1996 Senior Member of The Institute of Electrical and Electronics Engineers. His scientific interests are in fundamental research on Systems and Control Theory (realizations, analysis and control of nonlinear systems, modeling, digital control) and applied research in the field of spacecraft control. He served as coordinator of National Projects, member of the ASI (Italian Space Agency) Scientific and Technological Committees from 1990 to 1995 and 1997 to present; member of the Executive Council of the EUCA (European Union Control Association) from 1990 to 1997.

Dorothée Normand-Cyrot, from 1991 Dorothée Normand-Cyrot is Research Director at the CNRS laboratory of Signals and Systems in Gif-sur-Yvette, France. She is presently Director of a National CNRS Research Group in Systems Control Theory. Her research interest mainly concerns analysis and control of nonlinear discrete-time and sampled systems. Her more recent activity concerns the control of systems with sampled data, the development of multirate control strategies for stabilization, trajectory tracking, decoupling, motion planning and the representation of discrete-time dynamics as coupled differential and difference equations, thus allowing a hybrid study. Several applications are carried out in the domains of robotics, aeronautics, space and electrotechnology with industrial partners. D. Normand-Cyrot is author and co-author of almost 100 publications which include 51 journal papers and chapters of books, 43 published conference papers, various reports and two theses.