

Nonlinear discrete-time control of systems with a Naimark–Sacker bifurcation

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Abstract

In this paper we study the problem of the stabilization of nonlinear control system with one complex uncontrollable mode. We find normal forms and quadratic invariants; then we compute center manifolds, and use bifurcation theory to synthesize quadratic controllers. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Setting the study of dynamical systems in terms of specific normal forms is a powerful tool. Such an approach developed in both cases of dynamical systems defined by vector fields (differential dynamical systems) or maps (discrete-time systems), provides new coordinates under which the systems are transformed into their “simplest” forms [4,23,10].

The normal form approach was generalized to controlled dynamical systems in [21,19]. This is of peculiar interest since the normal form can be used for investigating stabilizability up to the computation of the stabilizing controller. In [19], with reference to a continuous-time system with controllable linear part, a controlled normal form was introduced as the simplest element of the equivalence class of a group of quadratic transformations performing both coordinates change and feedback. In discrete time, the approach was developed in [5].

The study is particularly attractive when dealing with nonlinear systems with uncontrollable modes. The idea was adopted in [6] to study nonlinear control systems with bifurcations through coordinates change without taking explicitly into account the control action. The investigation under the action of feedback has recently been proposed and developed in [1–3,8,9,11,14–16,20] for dynamics with one real or complex uncontrollable mode, thus providing a solution to the stabilization problem for nonlinear systems with bifurcations.

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The general stabilization procedure combines centre manifold techniques, normal forms and the use of a second-order static state feedbacks. In short, the linear part of the feedback is designed to stabilize the controllable subsystem while its quadratic part is designed for modifying the manifold over which the closed loop reduced dynamics evolves. Classical stability theorems for systems with bifurcations can be used to prove the stability of such a dynamics, under suitable conditions.

When dealing with controlled dynamical systems, it becomes difficult to parallel both the studies referred either to differential or difference nonlinear equations even if many analogies can be set (see [22]). This is due to the fact that the study of difference equations induces compositions of functions and the design of the control law always reduces to intricate problems of inversions of maps. The present paper addresses the problem of controlling nonlinear discrete-time control systems with one uncontrollable complex mode (systems with a Naimark–Sacker bifurcation). The case of one uncontrollable real mode has been studied in [20,13].

Section 2 is devoted to the computation of the normal form and its characterization in terms of invariants, a set of numbers which entirely specifies the normal form. In Section 3, we find the centre manifold and then solve the stabilization problem related to the Naimark–Sacker bifurcation by means of a quadratic controller.

2. Normal forms and invariants

Let us consider a single input, parameterized, nonlinear discrete-time dynamics on \mathbb{R}^n , described by the nonlinear difference equation

$$\zeta^+ = f(\zeta, \mu, u), \quad (1)$$

where $\zeta \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the input and $\mu \in \mathbb{R}$ the constant parameter, $f(\zeta, \mu, u) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n$ is analytic in its arguments. We adopt for any integer $k \geq 0$, the notation $\zeta^+ := \zeta(k+1) := f(\zeta(k), \mu, u(k))$. We assume that 0 is an equilibrium point associated with $\mu = 0$, $u = 0$ (i.e. $f(0, 0, 0) = 0$) and that the linear approximation around $(0, 0, 0)$, given by $\mathcal{A} := (\partial f / \partial \zeta)(0, 0, 0)$ and $\mathcal{B} := (\partial f / \partial u)(0, 0, 0)$ is such that

Assumption 1.

$$\text{rank}([\mathcal{B} \quad \mathcal{A}\mathcal{B} \quad \mathcal{A}^2\mathcal{B} \quad \dots \quad \mathcal{A}^{n-1}\mathcal{B}]) = n - 2, \quad (2)$$

with the uncontrollable part characterized by one complex mode, i.e., two conjugate eigenvalues $\lambda_{1,2}(0) = \cos(\omega) \pm j \sin(\omega)$ with ω such that $e^{kj\omega} \neq 1$ for $k = 1, 2, 3, 4$.

Remark. In the context of bifurcations for maps, eigenvalues on the unit disc are required not to be $+1$ or -1 or any of the square, cube or fourth roots of 1 to avoid what is called strong resonance [4,17].

2.1. Linear normal form

We first compute the linear normal form of (1) as it is essential to characterize the stability of the whole dynamics.

Lemma 2.1. *There exist a linear coordinates change and a feedback under which system (1), satisfying Assumption 1, takes the form*

$$\begin{aligned} \bar{z}^+ &= A_1 \bar{z} + \bar{f}_1^{[2]}(\bar{\zeta}, \mu) + \bar{g}_1^{[1]}(\bar{\zeta}, \mu)v + \bar{h}_1^{[0]}v^2 + O(\bar{\zeta}, \mu, v)^3, \\ \bar{x}^+ &= A_2 \bar{x} + B_2 v + \bar{f}_2^{[2]}(\bar{\zeta}, \mu) + \bar{g}_2^{[1]}(\bar{\zeta}, \mu)v + \bar{h}_2^{[0]}v^2 + O(\bar{\zeta}, \mu, v)^3, \end{aligned} \quad (3)$$

with $\tilde{\zeta} = [\tilde{z}^T, \tilde{x}^T]^T$, $\tilde{z} \in \mathbb{R}^2$, $\tilde{x} \in \mathbb{R}^{n-2}$,

$$A_1 = \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}_{(n-2) \times (n-2)}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(n-2) \times 1}, \quad (4)$$

$\tilde{f}_1^{[2]}, \tilde{f}_2^{[2]}$ (resp. $\tilde{g}_1^{[1]}, \tilde{g}_2^{[1]}$) stand for the homogeneous polynomials of degree 2 (resp. 1) while $\tilde{h}_1^{[0]}$ and $\tilde{h}_2^{[0]}$ are constants.

In (3), $O(\cdot)^3$ represents the remaining terms of order ≥ 3 in the expansion. In the sequel, for any analytic function A , for any $i \geq 0$, $A^{[i]}$ will denote the homogeneous i th-order polynomial.

Proof. From linear control theory [18], we know that there exist a linear coordinates change and a linear feedback, independent on μ , transforming (1) into

$$\tilde{z}^+ = A_1 \tilde{z} + D_1 \mu + O(\tilde{\zeta}, \mu, \tilde{v})^2,$$

$$\tilde{x}^+ = A_2 \tilde{x} + B_2 \tilde{v} + D_2 \mu + O(\tilde{\zeta}, \mu, \tilde{v})^2,$$

with $D_1 = [d_{11} \ d_{12}]^T$, $D_2 = [d_{21} \ d_{22} \ \cdots \ d_{2n-2}]^T$. Then, we easily recover (3) by choosing in addition

$$\tilde{z} = \tilde{z} - \frac{1}{2(1 - \cos(\omega))} \begin{bmatrix} -1 + \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & -1 + \cos(\omega) \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{12} \end{bmatrix} \mu,$$

$$\tilde{x} = \tilde{x} - \begin{bmatrix} 0 \\ d_{21} \\ \vdots \\ d_{2n-3} \end{bmatrix} \mu, \quad \tilde{v} = v - d_{2n-2} \mu. \quad \square$$

Remark. Let us point out some particular interests of the form exhibited in Lemma 2.1. The linear part does not depend on μ and the linearly uncontrollable part is separated from the controllable one which satisfies the canonical Brunovsky form up to the identity matrix. This choice which does not need any extra assumption is made to guarantee the local invertibility—invertibility of the linear approximation—of the dynamics (3) as will be justified later.

In the following, Σ_{NS} will denote the dynamics (3).

2.2. The quadratic normal form

Let us now consider *quadratic transformations* which are defined by the coupled action of a quadratic coordinates change and a quadratic static state feedback of the form,

$$\zeta = \tilde{\zeta} + \varphi^{[2]}(\tilde{\zeta}, \mu), \quad (5)$$

$$v = u + q^{[2]}(\tilde{\zeta}, \mu) + r^{[1]}(\tilde{\zeta}, \mu)u + s^{[0]}u^2, \quad (6)$$

where $\varphi^{[2]}, q^{[2]}, r^{[1]}, s^{[0]}$ denote homogeneous polynomials in their arguments. Such a transformation is employed to simplify the quadratic part of (3) while leaving the linear part unchanged.

Following [22] and because of the local invertibility of $\bar{f}(\bar{\zeta}, \mu, 0)$ (the linear part, $A = \text{diag}(A_1, A_2)$, is full rank), let us rewrite, the dynamics of Σ_{NS} , i.e. $\bar{\zeta}^+ = \bar{f}(\bar{\zeta}, \mu, v)$, in the exponential form

$$\bar{\zeta}^+ = e^{L_{\bar{G}_1^0(\bar{\zeta}, \mu)} + (v^2/2)L_{\bar{G}_2^0(\bar{\zeta}, \mu)}(Id)}|_{\bar{f}(\bar{\zeta}, \mu, 0)} + O(v)^3, \quad (7)$$

where by definition

$$\bar{G}^0(\bar{\zeta}, \mu, v) := \frac{\partial \bar{f}(\bar{\zeta}, \mu, v)}{\partial v} \Big|_{\bar{f}^{-1}(\bar{\zeta}, \mu, v)} = \sum_{i \geq 1} \frac{v^{i-1}}{i!} \bar{G}_i^0(\bar{\zeta}, \mu) = \bar{G}_1^0(\bar{\zeta}, \mu) + v\bar{G}_2^0(\bar{\zeta}, \mu) + O(v)^2.$$

As far as the notations are concerned we just note that for any analytic function $X(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $L_X := \tau(\cdot)\partial/\partial x|_x$, will denote the vector field associated with it and the right-hand side of (7) must be interpreted as the formal exponential,

$$e^{L_X} := I + L_X + \frac{1}{2}L_X^2 + \cdots + \frac{1}{k!}L_X^k + \cdots,$$

with $L_X^k := L_X \circ \cdots \circ L_X$ (k times). Moreover, for any $A(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, $L_X(A)|_x := \partial A(x)/\partial x|_x X(x)$.

Remark. The vector fields associated with the functions $\bar{G}_i^0(\bar{\zeta}, \mu)_{i \geq 1}$'s, used to set the exponential representation (7), are at the basis of a Lie approach in discrete time [22]. It will be shown in Section 2.3 that a family of invariants which characterize the quadratic normal form can be expressed in terms of these vector fields.

For studying the effect of the quadratic transformations (5), (6), it is enough to restrict the polynomial expansions of the dynamics (7) up to the order 2,

$$\bar{\zeta}^+ = \bar{f}(\bar{\zeta}, \mu, 0) + vL_{\bar{G}_1^0(\bar{\zeta}, \mu)}(Id)|_{\bar{f}(\bar{\zeta}, \mu, 0)} + \frac{v^2}{2}(L_{\bar{G}_1^0(\bar{\zeta}, \mu)} \circ L_{\bar{G}_1^0(\bar{\zeta}, \mu)} + L_{\bar{G}_2^0(\bar{\zeta}, \mu)})(Id)|_{\bar{f}(\bar{\zeta}, \mu, 0)} + O(v)^3,$$

with

$$\bar{f}(\bar{\zeta}, \mu, 0) := A\bar{\zeta} + \bar{f}^{[2]}(\bar{\zeta}, \mu) + O(\bar{\zeta}, \mu)^3,$$

$$\bar{G}_1^0(\bar{\zeta}, \mu) := \bar{G}_1^{0[0]} + \bar{G}_1^{0[1]}(\bar{\zeta}, \mu) + O(\bar{\zeta}, \mu)^2 = B + \bar{g}^{[1]}(A^{-1}\bar{\zeta}, \mu) + O(\bar{\zeta}, \mu)^2,$$

$$\bar{G}_2^0(\bar{\zeta}, \mu) + O(\bar{\zeta}, \mu) := \bar{G}_2^{0[0]} = 2\bar{h}^{[0]} + O(\bar{\zeta}, \mu),$$

where $B := [0, 0, B_2^T]^T$, $\bar{f}^{[2]} := [\bar{f}_1^{[2]T}, \bar{f}_2^{[2]T}]^T$, $\bar{g}^{[1]} := [\bar{g}_1^{[1]T}, \bar{g}_2^{[1]T}]^T$, and $\bar{h}^{[0]} := [\bar{h}_1^{[0]T}, \bar{h}_2^{[0]T}]^T$.

On these bases it is readily verified that

Proposition 2.1. *Under the quadratic transformations (5), (6) the dynamics (7) takes the form*

$$\zeta^+ = f(\zeta, \mu, 0) + uL_{G_1^0(\zeta, \mu)}(Id)|_{f(\zeta, \mu, 0)} + \frac{u^2}{2}(L_{G_1^0(\zeta, \mu)} \circ L_{G_1^0(\zeta, \mu)} + L_{G_2^0(\zeta, \mu)})(Id)|_{f(\zeta, \mu, 0)} + O(u)^3, \quad (8)$$

with

$$f(\zeta, \mu, 0) = \bar{f}(\zeta, \mu, 0) - A\varphi^{[2]}(\zeta, \mu) + \varphi^{[2]}(A\zeta, \mu) + q^{[2]}(\zeta, \mu)G_1^{0[0]} + O(\zeta, \mu)^3$$

$$= A\zeta + \bar{f}^{[2]}(\zeta, \mu) - A\varphi^{[2]}(\zeta, \mu) + \varphi^{[2]}(A\zeta, \mu) + q^{[2]}(\zeta, \mu)G_1^{0[0]} + O(\zeta, \mu)^3,$$

$$G_1^0(\zeta, \mu) = \bar{G}_1^0(\zeta, \mu) + L_{\bar{G}_1^0}(\varphi^{[2]}(\zeta, \mu)) + r^{[1]}(A^{-1}\zeta, \mu)\bar{G}_1^0 + O(\zeta, \mu)^2$$

$$= \bar{G}_1^{0[0]} + \bar{G}_1^{0[1]}(\zeta, \mu) + \frac{\partial \varphi^{[2]}(\zeta, \mu)}{\partial \zeta} \bar{G}_1^{0[0]} + r^{[1]}(A^{-1}\zeta, \mu)\bar{G}_1^{0[0]} + O(\zeta, \mu)^2,$$

$$G_2^{0[0]} = \bar{G}_2^{0[0]} + s^{[0]}\bar{G}_1^{0[0]} + O(\zeta, \mu). \quad (9)$$

Finally, the quadratic normal form is characterized in the next theorem.

Theorem 2.1 (Quadratic normal form). *There exist quadratic transformations (5), (6) under which (3) takes the form*

$$\zeta^+ = f^{[2]}(\zeta, \mu, 0) + uG_1^0 + \frac{u^2}{2} G_2^0 + O(\zeta, \mu, u)^3, \tag{10}$$

with

$$f^{[2]}(\zeta, \mu, 0) = \begin{bmatrix} A_1 z + \mu \beta z + x_1 \Gamma z + x_1 \mu \gamma^3 + \sum_{j=1}^{n-2} \delta^j x_j^2 \\ A_2 x + \sum_{j=1}^{n-2} \zeta^j x_j^2 \end{bmatrix},$$

$$G_1^0 := G_1^{0[0]} = B, \quad G_2^0 := G_2^{0[0]} = 2h^{[0]}. \tag{11}$$

The quadratic parts take the form

$$\mu \beta z = \mu \sum_{i=1}^2 \sum_{j=1}^2 \beta_i^j e_1^i z_j,$$

$$x_1 \Gamma z + x_1 \gamma^3 \mu = x_1 \sum_{i=1}^2 \left(\sum_{j=1}^2 \gamma_i^j z_j + \gamma_i^3 \mu \right) e_1^i,$$

$$\sum_{j=1}^{n-2} \delta^j x_j^2 = \sum_{i=1}^2 \sum_{j=1}^{n-2} \delta_i^j e_1^i x_j^2,$$

$$\sum_{j=1}^{n-2} \zeta^j x_j^2 = \sum_{i=1}^{n-2} \sum_{j=i+2}^{n-2} \zeta_i^j e_2^i x_j^2,$$

$$h^{[0]} = \begin{bmatrix} \sum_{i=1}^2 h_{1i}^{[0]} e_1^i \\ \sum_{i=1}^{n-2} h_{2i}^{[0]} e_2^i \end{bmatrix},$$

where β and Γ are in $\mathbb{R}^{2 \times 2}$; $\gamma^3, \delta^1, \delta^2, h_1^{[0]}$ are in $\mathbb{R}^{2 \times 1}$; $\zeta^j = [\zeta_1^j \dots \zeta_{n-2}^j]^\top, j = 1, \dots, n - 2, h_2^{[0]}$ are in $\mathbb{R}^{(n-2) \times 1}$, with constant entries $\beta_i^j, \gamma_i^j, \delta_i^j, \zeta_i^j, h_{ij}^{[0]}$ satisfying $\beta_1^1 = \beta_2^2, \beta_1^2 = -\beta_2^1, h_{2,n-2}^{[0]} = 0$ and $\zeta_i^j = 0; i = 1, \dots, j - 2. e_1^i$ (resp. e_2^j) denotes the i th-unit vector in the z -space (resp. x -space).

Remark. In (10), G_1^0 and G_2^0 are constant, $G_{1n-2}^{0[0]} = 1$ and the other vector fields are equal to 0, while $G_{2n-2}^{0[0]} = 0$.

The proof of Theorem 2.1 is detailed in [16]. We just give for completeness some aspects related to the exponential formulation.

Recalling that two quadratic dynamics with the same linear part are said to be quadratically equivalent if they are equal through quadratic transformations of the form (5), (6), we immediately deduce from (9) the homological equations below,

Proposition 2.2. *Two dynamics of the form (3), with the same linear part, described by the triplets $(f^{[2]}, G_1^{0[1]}, G_2^{0[0]})$ and $(\tilde{f}^{[2]}, \tilde{G}_1^{0[1]}, \tilde{G}_2^{0[0]})$, respectively, are quadratically equivalent if and only if there exist $(\varphi^{[2]}(\zeta, \mu), q^{[2]}(\zeta, \mu), r^{[1]}(\zeta, \mu), s^{[0]})$ satisfying the set of equations*

$$\begin{aligned} \tilde{f}^{[2]}(\zeta, \mu, 0) - f^{[2]}(\zeta, \mu, 0) &= -A\varphi^{[2]}(\zeta, \mu) + \varphi^{[2]}(A\zeta, \mu) + q^{[2]}(\zeta, \mu)G_1^{0[0]}, \\ \tilde{G}_1^{0[1]}(\zeta, \mu) - G_1^{0[1]}(\zeta, \mu) &= L_{G_1^{0[0]}}(\varphi^{[2]}(\zeta, \mu)) + r^{[1]}(A^{-1}\zeta, \mu)G_1^{0[0]}, \\ \tilde{G}_1^{0[0]} - G_1^{0[0]} &= 0, \\ \tilde{G}_2^{0[0]} - G_2^{0[0]} &= s^{[0]}G_1^{0[0]}. \end{aligned} \quad (12)$$

Eqs. (12) are at the basis of the proof of Theorem 2.1. They can be referred to as the *homological equations* thus extending the usual terminology to controlled dynamics. The normal form corresponds to the remaining terms once all the possible quadratic terms have been simplified by appropriately choosing $(\varphi^{[2]}(\zeta, \mu), q^{[2]}(\zeta, \mu), r^{[1]}(\zeta, \mu), s^{[0]})$. More precisely, to do so, one computes the image set of a certain linear map—defined by the first equation in (12)—subject to constraints—defined by the last three equations in (12). Then, the complementary space contains the terms which cannot be simplified, precisely those which characterize the normal form. In general, several normal forms can be built depending on the basis chosen for the complementary space. The structure of (10) has been chosen because, for $(x, u) = (0, 0)$, the z -part exactly restores the Poincaré normal form

$$z^+ = \left(A_1 + \begin{bmatrix} \beta_1^1 & \beta_1^2 \\ -\beta_1^2 & \beta_1^1 \end{bmatrix} \mu \right) z,$$

which exhibits a Naimark–Sacker bifurcation [23,10], when β_1^1 and β_1^2 are such that the condition of Theorem 3.2 holds true (Section 3.2 in the sequel).

2.3. The quadratic invariants

We show in this section that the quadratic normal form (10) can be exclusively characterized by the *the quadratic invariants*, a set of numbers which remain unchanged under quadratic transformations. Moreover, the coefficients of the quadratic normal form can be uniquely determined in terms of these numbers. They generalize the *characteristic numbers* introduced in [19] to the present situation, for the continuous-time case, and [5] for the discrete-time one, with reference to controllable dynamics.

Hereafter, we will show that these numbers can be expressed in terms of the $(G_i^0(\zeta, \mu), i = 1, 2)$. To do so, as in [22], let us denote iteratively by $G_i^{j+1}(\zeta, \mu)$, $i = 1, 2$, $j \geq 0$, the transport of any vector field $G_i^j(\zeta, \mu)$ along the drift $f(\zeta, \mu, 0)$,

$$G_i^{j+1}(\zeta, \mu) := \left(\frac{\partial f(\zeta, \mu, 0)}{\partial \zeta} G_i^j(\zeta, \mu) \right) \Big|_{f^{-1}(\zeta, \mu, 0)}. \quad (13)$$

For any analytic vector fields $X(x)$ and $Y(x)$ defined in \mathbb{R}^n $ad_X(Y) := [X, Y] := L_X \circ L_Y - L_Y \circ L_X := (\partial Y / \partial x)X - (\partial X / \partial x)Y$.

Denoting by C_x , C_z , X_z and X_μ the matrices

$$C_x = [0 \quad 0 \quad I_{n-2}]_{(n-2) \times n}, \quad C_z = [I_2 \quad 0 \quad \cdots \quad 0]_{2 \times n}, \quad X_z = \begin{bmatrix} I_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 2}, \quad X_\mu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{1 \times n},$$

where I_v are the identity matrices in $\mathbb{R}^{v \times v}$, we set

Definition 2.1. Given the dynamics (3) (or equivalently (7)), let us define the following list of numbers:

$$\begin{aligned}
 b_1^t &= C_z[G_1^0, G_1^t]^{[0]}, \quad 1 \leq t \leq 2, \\
 a_1^t &= C_x[G_1^0, G_1^t]^{[0]}, \quad 1 \leq t \leq n-2, \\
 h_1^{[0]} &= \frac{1}{2} C_z G_2^{[0]}, \\
 h_2^{[0]} &= \frac{1}{2} C_x G_2^{[0]}, \\
 \Gamma &= \begin{bmatrix} \gamma_1^1 & \gamma_1^2 \\ \gamma_2^1 & \gamma_2^2 \end{bmatrix} = C_z \frac{\partial G_1^{n-2[1]}(\zeta, \mu)}{\partial z}, \\
 \gamma^3 &= \begin{bmatrix} \gamma_1^3 \\ \gamma_2^3 \end{bmatrix} = C_z \frac{\partial G_1^{n-2[1]}(\zeta, \mu)}{\partial \mu}, \\
 \beta &= \begin{bmatrix} \beta_1^1 & \beta_1^2 \\ -\beta_1^2 & \beta_1^1 \end{bmatrix} = C_z \frac{\partial^2 f^{[2]}(\zeta, \mu, 0)}{\partial z \partial \mu},
 \end{aligned} \tag{14}$$

where $f^{[2]}, G_1^t, G_2$ are defined by (11), (13).

The exponential representation enables us to deduce that the vectors of numbers a_1^t, b_1^t are exactly the coefficients of the product of inputs $u(t)u(0)$ in the quadratic expansion of the state $\zeta(t+1)$ at time $t+1$, when considered as a functional in the sequence of inputs from time 0 up to time t . The numbers a_1^t, b_1^t are directly linked to the entries of the matrices ζ, δ in (10).

The following theorem specifies their properties as invariants:

Theorem 2.2. Consider the dynamics (3) (or equivalently (7)), then

- (i) The list of numbers (14) do not change under quadratic transformations of the form (5), (6).
- (ii) The list of numbers (14) are uniquely associated with the coefficients of the quadratic terms in the normal form (10).
- (iii) Given two dynamics of the form (3), with the same linearization (same ω), they are quadratically equivalent if and only if the lists of numbers (14) are equal.

The numbers (14) will be denoted as quadratic invariants.

Proof. (i) From (12), it becomes clear that quadratic transformations (5), (6) modify the quadratic part of the drift $f(\cdot, 0)$, the linear part of $G_1^0(\cdot)$ and the constant part of G_2^0 only. With reference to the invariance of the a_1^t, b_1^t , it is an immediate consequence of the form of the transformations and the remark above. With reference to the invariance of $h_2^{[0]}$, it directly follows from (12) since $h_{2n-2}^{[0]} = 0$. With reference to the elements of Γ, γ^3, β , it is a matter of computations to verify that, because of the definition of G_1^{n-2} , we get

$$C_z \frac{\partial G_1^{n-2[1]}(\zeta, \mu)}{\partial z} = C_z \frac{\partial^2 f_{n-2}^{[2]}(\zeta, \mu, 0)}{\partial z \partial x_{n-2}} = \Gamma$$

and

$$C_z \frac{\partial G_1^{n-2[1]}(\zeta, \mu)}{\partial \mu} = C_z \frac{\partial^2 f_{n-2}^{[2]}(\zeta, \mu, 0)}{\partial \mu \partial x_{n-2}} = \gamma^3,$$

where $f_{n-2}^{[2]}(\zeta, \mu, 0)$ denotes the quadratic part of the composition $n-2$ times of f . As $\varphi^{[2]}(\zeta, \mu)$ is quadratic, any second-order derivative is constant and invariance follows since,

$$C_z \frac{\partial^2 \tilde{f}^{n-2[2]}}{\partial x_{n-2} \partial z} = C_z \frac{\partial^2 f_{n-2}^{[2]}}{\partial x_{n-2} \partial z} + A_1 \frac{\partial^2 \varphi^{[2]}}{\partial x_{n-2} \partial z} - C_z \frac{\partial^2 \varphi^{[2]}}{\partial x_{n-2} \partial z} \Big|_{A\tilde{\zeta}} = C_z \frac{\partial^2 f_{n-2}^{[2]}}{\partial x_{n-2} \partial z}$$

and noting that $q^{[2]}$ affects the last x_{n-2} component only.

The same argumentation is valid for $C_z \partial^2 f_{n-2}^{[2]}(\zeta, \mu, 0) / \partial \mu \partial x_{n-2} = \gamma^3$, and $C_z \partial^2 f^{[2]}(\zeta, \mu) / \partial z \partial \mu = \beta$.

(ii) The proof is done by induction showing that the vectors δ^ℓ (resp. ζ^ℓ) for $\ell = 1, \dots, n-2$ are suitable linear combinations of the b_1^ℓ, a_1^ℓ . From

$$G_1^k := \frac{\partial f(\zeta, \mu, 0)}{\partial \zeta} G_1^{k-1} \Big|_{A^{-1}\zeta}$$

we immediately verify for $(k = 1, \dots, n-2)$, that $G_1^{k,[0]}$ is the last column of the matrix A_2^k completed with 0 with $(A_2^k)_{i,n-2} = C_k^{n-2-i} = k! / (n-2-i)! (k-n+2+i)!$ for $(i = n-2-k, \dots, n-2)$. Easy but tedious computations enable us to further deduce the following formulae:

$$C_z G_1^k = (A_1 C_z G_1^{k-1} + 2[0, 0, \delta^1 x_1, \dots, \delta^{n-2} x_{n-2}] G_1^{k-1,[0]}) \Big|_{A_2^{-1}x},$$

$$C_x G_1^k = (A_2 C_x G_1^{k-1} + 2[0, 0, \zeta^1 x_1, \dots, \zeta^{n-2} x_{n-2}] G_1^{k-1,[0]}) \Big|_{A_2^{-1}x},$$

where $(A_2^{-1}x)_i = \sum_{j=0}^{n-1-i} (-1)^j x_{i+j}$.

Moreover, since $G_1^0 = G_1^{0,[0]} = B = [0, \dots, 0, 1]^T$, it follows that

$$b_1^t := C_z [G_1^0, G_1^t]^{[0]} = \frac{\partial C_z G_1^{t[1]}(\zeta, \mu)}{\partial x_{n-2}} \quad \text{and} \quad a_1^t := C_x [G_1^0, G_1^t]^{[0]} = \frac{\partial C_x G_1^{t[1]}(\zeta, \mu)}{\partial x_{n-2}}.$$

Further computations enable us to show that, for $r < t$,

$$C_z [G_1^r, G_1^t]^{[0]} := C_z A^r [G_1^0, G_1^{t-r}]^{[0]} = A_1^r C_z [G_1^0, G_1^{t-r}]^{[0]},$$

$$C_x [G_1^r, G_1^t]^{[0]} := C_x A^r [G_1^0, G_1^{t-r}]^{[0]} = A_2^r C_x [G_1^0, G_1^{t-r}]^{[0]}.$$

According to this, we immediately get for the first terms

$$\frac{\partial C_z G_1^{1[1]}(\zeta, \mu)}{\partial x_{n-2}} = 2\delta^{n-2} = b_1^1 \quad \text{and} \quad \frac{\partial C_x G_1^{1[1]}(\zeta, \mu)}{\partial x_{n-2}} = 2\zeta^{n-2} = a_1^1.$$

Working out the relations (15) for $k \geq 2$, we easily notice that the first occurrence of δ^{n-1-k} (resp. ζ^{n-1-k}) is in $C_z G_1^k$ (resp. $C_x G_1^k$) and that the coefficients δ^j (resp. ζ^j) are linear combinations of the $A_1^r b_1^{t-r}$ (resp. $A_2^r a_1^{t-r}$).

Let us illustrate this for the second-order terms. From (15), we get the equalities

$$C_z [G_1^0, G_1^2]^{[0]} := \frac{\partial C_z G_1^{2[1]}}{\partial x_{n-2}} = 2A_1 \delta^{n-2} + 2\delta^{n-2} - 2\delta^{n-3} = b_1^2,$$

$$C_x [G_1^0, G_1^2]^{[0]} := \frac{\partial C_x G_1^{2[1]}}{\partial x_{n-2}} = 2A_2 \zeta^{n-2} + 2\zeta^{n-2} - 2\zeta^{n-3} = a_1^2,$$

with $A_1 C_z [G_1^0, G_1^1]^{[0]} = C_z [G_1^1, G_1^2]^{[0]} = A_1 b_1^1$ and $A_2 C_x [G_1^0, G_1^1]^{[0]} = C_x [G_1^1, G_1^2]^{[0]} = A_2 a_1^1$, thus obtaining

$$2\delta^{n-3} = A_1 C_z [G_1^0, G_1^1]^{[0]} + C_z [G_1^0, G_1^1]^{[0]} - C_z [G_1^0, G_1^2]^{[0]} = A_1 b_1^1 + b_1^1 - b_1^2,$$

$$2\zeta^{n-3} = A_2 C_x [G_1^0, G_1^1]^{[0]} + C_x [G_1^0, G_1^1]^{[0]} - C_x [G_1^0, G_1^2]^{[0]} = A_2 a_1^1 + a_1^1 - a_1^2.$$

For the elements quadratic in u we immediately get

$$h_1^{[0]} = b_2^0 = \frac{1}{2} C_z G_2^{0[0]}, \quad h_2^{[0]} = a_2^0 = \frac{1}{2} C_x G_2^{0[0]}.$$

(iii) is immediate recalling that two systems with the same linear part are said quadratically equivalent if they have the same quadratic normal forms and thus the same invariants. \square

3. Control design

The general procedure generalizes to controlled dynamics, the one combining the centre manifold theorem and normal forms proposed in the literature in both continuous-time and discrete-time contexts with reference to dynamical systems (see [23,10,12] and the references therein). It combines centre manifold techniques, normal form and the use of second-order static state feedback to achieve stabilizability conditions. Briefly speaking, the linear part of the feedback is designed to stabilize the controllable x -part. Then, the basic idea is to suitably modify through the quadratic part of the feedback the manifold equation so as to ensure stability of the closed loop reduced dynamics. Classical stability theorems of systems with bifurcations can be used to conclude. It has to be noted that the control law is designed by solving a matrix inequality depending on the quadratic invariants.

Let the overall quadratic stabilizing feedback be

$$u(\zeta, \mu) = R_1 z + R_2 x + R_3 \mu + [z \quad \mu] Q_{fb} \begin{bmatrix} z \\ \mu \end{bmatrix} + O(\zeta, \mu)^3, \tag{15}$$

where $R_1 := [r_1^1, r_2^1]$, $R_2 := [r_1^2, \dots, r_{n-2}^2]$, $R_3 := r_1^3$, are row vectors with constant entries ($r_1^2 \neq 0$), and Q is a square matrix of order 3.

First, the coefficients r_i^2 stabilize the x -part, while the stabilization of the bidimensional z -dynamics is achieved through centre manifold techniques [7]. More precisely, let R_2 be such that the matrix $A_2 + B_2 R_2$ has eigenvalues inside the unit disc. Due to the structure of A_2 and B_2 , $A_2 + B_2 R_2$ exhibits the canonical form

$$A_2 + B_2 R_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 & \dots & 1 + r_{n-2}^2 \end{bmatrix},$$

its characteristic polynomial is $\mathcal{P}(\lambda) := (\lambda - 1)^{n-2} - \sum_{i=1}^{n-2} r_i^2 (\lambda - 1)^{i-1}$.

3.1. Computation of the centre manifold

Let us determine the equations of the centre manifold $x = \Pi(z, \mu)$. We first show in Proposition 3.1 that the linear part in (z, μ) of the feedback (i.e. R_1, R_3) can be neglected.

Proposition 3.1. *Given the linear closed loop dynamics*

$$\begin{aligned} z^+ &= A_1 z, \\ x^+ &= A_2 x + B_2 (R_1 z + R_2 x + R_3 \mu), \end{aligned}$$

there exists a linear coordinates change under which the (z, μ) -terms are removed from the x -dynamics thus getting

$$\begin{aligned} \tilde{z}^+ &= A_1 \tilde{z}, \\ \tilde{x}^+ &= A_2 \tilde{x} + B_2 R_2 \tilde{x}. \end{aligned} \tag{16}$$

Proof. Just set

$$\tilde{z} = z,$$

$$\tilde{x}_1 = x_1 + [\varphi_1 \quad \varphi_2]z + \varphi_3\mu,$$

$$\tilde{x}_2 = x_2 + [\varphi_1 \quad \varphi_2](A_1 - Id_2)z,$$

\vdots

$$\tilde{x}_{n-2} = x_{n-2} + [\varphi_1 \quad \varphi_2](A_1 - I_2)^{n-3}z,$$

with $\varphi_3 = R_3/r_1^2$, $[\varphi_1 \quad \varphi_2] = -R_1\mathcal{P}^{-1}(A_1 - I_2)$. \square

Applying now the feedback (15) to the normal form (10), we write the centre manifold equation which turns out to depend only on the quadratic part of the feedback. Due to Proposition 3.1, we will assume below $R_1 = 0$ and $R_3 = 0$ in (15).

The next theorem gives the explicit expression of the centre manifold.

Theorem 3.1. *Given the quadratic feedback (15) then, in the coordinates defined in Proposition 3.1, the centre manifold results to be quadratic with the i th-component given by*

$$\Pi_i^{[2]}(z, \mu) = [z \quad \mu](-1)^{i-1} \widetilde{Ad}_{\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix}}^{i-1}(\mathcal{Q}_i) \begin{bmatrix} z \\ \mu \end{bmatrix}, \quad (17)$$

with \mathcal{Q}_{fb} directly linked to \mathcal{Q}_i (through Eq. (20) hereafter), and $\widetilde{Ad}_X(Y) := X - Y^TXY$ for all square matrices X and Y .

Proof. Consider the normal form (10) and the feedback (15) and assume the centre manifold to be

$$x = \Pi(z, \mu) = \Pi^{[1]} \begin{bmatrix} z \\ \mu \end{bmatrix} + [z \quad \mu] \mathcal{Q} \begin{bmatrix} z \\ \mu \end{bmatrix} + O(z, \mu)^3,$$

with i th component

$$x_i = \Pi_i^{[1]} \begin{bmatrix} z \\ \mu \end{bmatrix} + [z \quad \mu] \mathcal{Q}_i \begin{bmatrix} z \\ \mu \end{bmatrix},$$

where for $i = 1, \dots, n-2$, the \mathcal{Q}_i are real 3×3 symmetric matrices, and $\Pi_i^{[1]} = [\Pi_{i,1}^{[1]} \quad \Pi_{i,2}^{[1]} \quad \Pi_{i,3}^{[1]}]$.

From the centre manifold equation $x^+ = \Pi(z^+, \mu)$, we get the equality

$$\begin{aligned} (A_2 + B_2R_2) \left(\Pi^{[1]} \begin{bmatrix} z \\ \mu \end{bmatrix} + [z \quad \mu] \mathcal{Q} \begin{bmatrix} z \\ \mu \end{bmatrix} \right) + [A_1z \quad \mu] \mathcal{Q}_{fb} \begin{bmatrix} A_1z \\ \mu \end{bmatrix} \\ = \left(\Pi^{[1]} + [z \quad \mu] \begin{bmatrix} A_1^T & 0 \\ 0 & 1 \end{bmatrix} \mathcal{Q} \right) \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ \mu \end{bmatrix}. \end{aligned}$$

After easy computations performed on each component, it results that the linear part of the centre manifold satisfies the conditions

$$\Pi_{i+1}^{[1]} + \Pi_i^{[1]} = \Pi_i^{[1]} \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Pi_{n-2}^{[1]} + \sum_{i=1}^{n-2} r_i^2 \Pi_i^{[1]} + [R_1 \quad R_3] = \Pi_{n-2}^{[1]} \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the quadratic part satisfies the conditions

$$\begin{aligned} \mathcal{Q}_{i+1} &= -\widetilde{Ad}_{\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix}}(\mathcal{Q}_i) - h_{2,i}^{[0]} \left(\Pi_i^{[1]T} \Pi_i^{[1]} + \begin{bmatrix} R_1 \\ R_3 \end{bmatrix} [R_1 \quad R_3] + \sum_{j=i+2}^{n-2} \zeta_i^j \Pi_i^{[1]T} \Pi_i^{[1]} \right), \\ \mathcal{Q}_{fb} &= \mathcal{Q}_{n-2} - \sum_{i=1}^{n-2} r_i^2 \mathcal{Q}_i - \widetilde{Ad}_{\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix}}(\mathcal{Q}_{n-2}). \end{aligned} \tag{18}$$

In conclusion, we get for $i \leq n - 3$

$$\begin{aligned} \Pi_{i+1}^{[1]} &= \Pi_1^{[1]} \begin{bmatrix} A_1 - I_2 & 0 \\ 0 & 0 \end{bmatrix}^i, \\ \sum_{i=1}^{n-2} r_i^2 \Pi_1^{[1]} \begin{bmatrix} A_1 - I_2 & 0 \\ 0 & 0 \end{bmatrix}^{i-1} + [R_1 \quad R_3] &= \Pi_1^{[1]} \begin{bmatrix} A_1 - I_2 & 0 \\ 0 & 0 \end{bmatrix}^{n-2}, \end{aligned}$$

so that

$$\begin{aligned} \Pi_1^{[1]} \mathcal{P} \left(\begin{bmatrix} A_1 - I_2 & 0 \\ 0 & 0 \end{bmatrix} \right) &= [R_1 \quad R_3], \\ \Pi_{i+1}^{[1]} &= \Pi_1^{[1]} \begin{bmatrix} A_1 - I_2 & 0 \\ 0 & 0 \end{bmatrix}^i. \end{aligned}$$

For $r_1^2 \neq 0$, and from the expression of $\mathcal{P}(\lambda)$ we finally obtain

$$\begin{aligned} \Pi_1^{[1]} &= [R_1 \quad R_3] \begin{bmatrix} P(A_1 - I_2)^{-1} & 0 \\ 0 & -\frac{1}{r_1^2} \end{bmatrix}, \\ \Pi_{i+1}^{[1]} &= \Pi_1^{[1]} \begin{bmatrix} A_1 - I_2 & 0 \\ 0 & 0 \end{bmatrix}^i. \end{aligned} \tag{19}$$

Recalling (Proposition 3.1) the possibility of rendering the linear part of the feedback independent on z and μ (i.e. $R_1 = R_3 = 0$), it follows from (19) that the linear part of the centre manifold can be removed, i.e. $\Pi_i^{[1]} = 0$, for $i = 1, \dots, n - 2$. This reduces (18) to

$$\begin{aligned} \mathcal{Q}_{i+1} &= (-1)^i \widetilde{Ad}_{\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix}}^i(\mathcal{Q}_1), \\ \mathcal{Q}_{fb} &= \begin{bmatrix} A_1^T & 0 \\ 0 & 1 \end{bmatrix} \mathcal{Q}_{n-2} \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix} - \sum_{i=1}^{n-2} r_i^2 \mathcal{Q}_i. \quad \square \end{aligned} \tag{20}$$

Theorem 3.1 implies that, for any given matrix \mathcal{Q}_1 , there always exists \mathcal{Q}_{fb} , given by (20), so that the associated feedback (15) yields a centre manifold satisfying (17).

3.2. Stabilization of Σ_{NS}

In this section, we will give sufficient conditions for the stabilization of Σ_{NS} and we will design a controller. Owing to the Poincaré–Andronov–Hopf theorem for maps, hereafter recalled, it will be proved that the stability of the Naimark–Sacker bifurcation is determined by \mathcal{Q}_1 and the invariants.

Theorem 3.2 (Guckenheimer and Holmes [12]). *Consider the dynamics in \mathbb{R}^2*

$$\begin{bmatrix} z_1^+ \\ z_2^+ \end{bmatrix} = \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \Psi(z_1, z_2, \mu) \\ \tilde{\Psi}(z_1, z_2, \mu) \end{bmatrix},$$

with $e^{kj\omega} \neq 1$, for $k = 1, 2, 3, 4$; then if

$$\begin{aligned} \vartheta &= \cos(\omega)(\Psi_{\mu z_1} + \tilde{\Psi}_{\mu z_2}) + \sin(\omega)(\tilde{\Psi}_{\mu z_1} - \Psi_{\mu z_2}) \neq 0, \\ \check{\alpha} &\neq 0, \end{aligned}$$

where $\check{\alpha}$ is a constant as defined below, an invariant manifold bifurcates into either $\mu > 0$ or $\mu < 0$, depending on the signs of the non-zero expressions above. This invariant manifold, diffeomorphic to a circle, is attracting if it bifurcates into the region of μ for which the region is unstable (a supercritical bifurcation) and repelling if it bifurcates into the region for which the origin is stable (a subcritical bifurcation).

$\check{\alpha}$ is a constant given by

$$\check{\alpha} = -\Re \left[\frac{(1 - 2e^{j\omega})e^{-2j\omega}}{1 - e^{j\omega}} \zeta_{11} \zeta_{20} \right] - \frac{1}{2} (|\zeta_{11}|^2 - |\zeta_{02}|^2) + \Re(e^{-j\omega} \zeta_{21}),$$

with

$$\zeta_{20} = \frac{1}{8} \{ \psi_{z_1 z_1} - \psi_{z_2 z_2} + 2\tilde{\psi}_{z_1 z_2} + j(\tilde{\psi}_{z_1 z_1} - \tilde{\psi}_{z_2 z_2} - 2\psi_{z_1 z_2}) \},$$

$$\zeta_{11} = \frac{1}{4} \{ \psi_{z_1 z_1} + \psi_{z_2 z_2} + j(\tilde{\psi}_{z_1 z_1} + \tilde{\psi}_{z_2 z_2}) \}$$

$$\zeta_{02} = \frac{1}{8} \{ \psi_{z_1 z_1} - \psi_{z_2 z_2} - 2\tilde{\psi}_{z_1 z_2} + j(\tilde{\psi}_{z_1 z_1} - \tilde{\psi}_{z_2 z_2} + 2\psi_{z_1 z_2}) \}$$

$$\zeta_{21} = \frac{1}{16} \{ \psi_{z_1 z_1 z_1} + \psi_{z_1 z_2 z_2} + \tilde{\psi}_{z_1 z_1 z_2} + \tilde{\psi}_{z_2 z_2 z_2} + j(\tilde{\psi}_{z_1 z_1 z_1} + \tilde{\psi}_{z_1 z_2 z_2} - \psi_{z_1 z_1 z_2} - \psi_{z_2 z_2 z_2}) \},$$

where $\Re(\cdot)$ stands for the real part of the function in the parentheses. For any analytic function $\Lambda(x)$, the partial derivatives, denoted by $\Lambda_{x_i x_j}(x) := \partial^2 \Lambda(x) / \partial x_i \partial x_j$, are evaluated at the bifurcation point, i.e. $(z, \mu) = (0, 0)$.

Substituting (17) into the z^+ dynamics of the normal form (10), one computes the critical coefficients $\check{\alpha}$, ϑ , we have

$$\begin{aligned} \check{\alpha} &= \frac{1}{8} \{ ((3\gamma_1^1 + \gamma_2^2) \cos(\omega) + (3\gamma_2^1 - \gamma_1^2) \sin(\omega)) \mathcal{Q}_{1;11} + ((\gamma_1^1 + 3\gamma_2^2) \cos(\omega) + (\gamma_2^2 - \gamma_1^1) \sin(\omega)) \mathcal{Q}_{1;22} \\ &\quad + ((\gamma_1^2 + \gamma_2^1) \cos(\omega) + (\gamma_2^2 - \gamma_1^1) \sin(\omega)) (\mathcal{Q}_{1;12} + \mathcal{Q}_{1;21}) + \Gamma_z \cos(\omega) + \tilde{\Gamma}_z \sin(\omega) \}, \\ \vartheta &= 2(\beta_1^1 \cos(\omega) - \beta_1^2 \sin(\omega)), \end{aligned} \tag{21}$$

where

$$\mathcal{Q}_1 = \begin{bmatrix} \mathcal{Q}_{1;11} & \mathcal{Q}_{1;12} & \mathcal{Q}_{1;13} \\ \mathcal{Q}_{1;21} & \mathcal{Q}_{1;22} & \mathcal{Q}_{1;23} \\ \mathcal{Q}_{1;31} & \mathcal{Q}_{1;32} & \mathcal{Q}_{1;33} \end{bmatrix}$$

represents the first component of the centre manifold (set $i = 1$ in (17)). It is directly linked to the quadratic feedback through (20). Moreover, Γ_z and $\tilde{\Gamma}_z$ are given by

$$\Gamma_z = \frac{\partial^3 f_{11}^{[3]}}{\partial z_1^3} + \frac{\partial^3 f_{11}^{[3]}}{\partial z_1 \partial z_2^2} + \frac{\partial^3 f_{12}^{[3]}}{\partial z_1^2 \partial z_2} + \frac{\partial^3 f_{12}^{[3]}}{\partial z_2^3},$$

$$\tilde{\Gamma}_z = \frac{\partial^3 f_{12}^{[3]}}{\partial z_1^3} + \frac{\partial^3 f_{12}^{[3]}}{\partial z_1 \partial z_2^2} - \frac{\partial^3 f_{11}^{[3]}}{\partial z_1^2 \partial z_2} - \frac{\partial^3 f_{11}^{[3]}}{\partial z_2^3},$$

with

$$f_1^{[3]} = \begin{bmatrix} f_{11}^{[3]} \\ f_{12}^{[3]} \end{bmatrix},$$

the cubic part of the z^+ -equation in (10).

The next theorem is a straightforward corollary of Theorem 3.2.

Theorem 3.3. *Given a discrete-time dynamics in the normal form (10), suppose $\vartheta \neq 0$. If one of the following conditions is satisfied:*

- $\Gamma_1 = (3\gamma_1^1 + \gamma_2^2) \cos(\omega) + (3\gamma_2^1 - \gamma_1^2) \sin(\omega) \neq 0$,
- $\Gamma_2 = (\gamma_1^1 + 3\gamma_2^2) \cos(\omega) + (\gamma_2^2 - \gamma_1^1) \sin(\omega) \neq 0$,
- $\Gamma_3 = (\gamma_1^2 + \gamma_2^1) \cos(\omega) + (\gamma_2^2 - \gamma_1^1) \sin(\omega) \neq 0$,

there always exists a nonlinear feedback (15) that makes the Naimark–Sacker bifurcation supercritical (resp. subcritical). Q is built according to (20), with Q_1 chosen to satisfy $\check{\alpha} < 0$ (resp. $\check{\alpha} > 0$).

In conclusion, in the coordinate system given by Proposition 3.1, the stabilizing controller¹ is given by (15)

$$u(\tilde{\zeta}, \mu) = R_2 \tilde{x} + [z \quad \mu] Q_{fb} \begin{bmatrix} z \\ \mu \end{bmatrix} + O(\zeta, \mu)^3, \tag{22}$$

with $A_2 + B_2 R_2$ Hurwitz, and Q_{fb} given by (20). More precisely, when a supercritical bifurcation is required, we have to solve the following inequality:

$$\check{\alpha} = \frac{1}{8} \{ \Gamma_1 \mathcal{Q}_{1;11} + \Gamma_2 \mathcal{Q}_{1;22} + \Gamma_3 (\mathcal{Q}_{1;12} + \mathcal{Q}_{1;21}) + \Gamma_z \cos(\omega) + \tilde{\Gamma}_z \sin(\omega) \} < 0.$$

Suppose $\Gamma_1 \neq 0$. A particular solution to this inequality is given by

$$\mathcal{Q}_{1;11} = - \frac{\Gamma_z \cos(\omega) + \tilde{\Gamma}_z \sin(\omega)}{\Gamma_1}, \quad \mathcal{Q}_{1;22} = - \Gamma_2, \quad \mathcal{Q}_{1;12} = \mathcal{Q}_{1;21} = - \frac{\Gamma_3}{2}.$$

The other cases are treated similarly.

4. Conclusion

The stabilization of a discrete-time controlled dynamics with one complex uncontrollable mode is studied using quadratic normal forms, centre manifold techniques and quadratic feedbacks. A procedure for designing a quadratic controller is proposed. The present work can be considered as the discrete-time counterpart of [14]. Since Hopf bifurcations are transformed under sampling into the Naimark–Sacker bifurcations, it should be possible to study supercritical or subcritical conditions over bifurcations under sampling. Work is progressing in these directions.

¹ R_1 and R_3 could be chosen to be equal to zero.

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