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A link between input–output stability and Lyapunov stability[1]

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Abstract

It is shown that incremental boundedness of an input–output operator ensures global asymptotic stability of any motion. A reciprocal statement is set.

Keywords: Incremental boundedness; Lyapunov stability

1. Introduction

In this note, a link between input–output stability and Lyapunov stability is given using the notion of incremental gain. A well-known result of Willems examines in a nonlinear context the links between finite gain stability and Lyapunov stability [16]. More precisely, in [16] it is proved that under uniform observability and reachability conditions, L_2-gain stability ensures uniform asymptotic stability of the equilibrium point associated with the null control. In [8], a local version of this result is obtained in the context of multiple equilibrium points. In [5], it is shown that incremental boundedness ensures asymptotic stability of the equilibria which are associated with any constant control.

Hereafter, considering a nonlinear dynamical system generated by differential equations, it is shown that incremental boundedness ensures asymptotic stability of any motion [7,21]. More precisely, incremental boundedness ensures that under perturbations of the initial state and for any prefixed input not necessarily constant the output and state evolution asymptotically converge to the unperturbed ones. Finally, a reciprocal statement is given showing that uniform asymptotic stability assumptions lead to incremental boundedness.

In fact, these results precise an intuitive concept, not known in our opinion, concerning the role played by input/output continuity in a nonlinear context [4, 6].

After recalling some classical notations and definitions in Section 2, the results are stated in Section 3.

2. Notations

The notations and terminology, which are recalled hereafter, are classical in an input–output context [12, 20, 3, 8]. Denoting by E, the set of real measurable n vector valued functions of the real variable t, on R^+, one defines $L_2^n = (x \in E \mid \|x\|_2 < \infty)$ where $\|x\|_2 = \sqrt{\int_0^\infty x(t)^{\mathrm{T}} x(t)\,\mathrm{d}t}$

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and $L_2^{n,e} = \{x \mid x \in E \mid P_\tau x \in L_2^n, \forall \tau \in R^+\}$, its associated extended space, where P_τ is the causal operator which truncates a signal at time τ. For the sake of simplicity, one sets $\|u\|_{2,\tau} \triangleq \|P_\tau u\|_2$.

Definition 2.1 (Desoer and Vidyasagar [3] and Hill and Moylan [8]). An operator H, defined from $L_2^{m,e}$ into $L_2^{p,e}$, is *weakly L_2-gain* stable if there exist finite non-negative constants γ and β such that $\|H(u)\|_2 \leqslant \gamma\|u\|_2 + \beta$ for all $u \in L_2^m$. Its gain coincides with the minimum value of γ. When $\beta = 0$, the system is said to be *L_2-gain stable* and $\|H\|_{i_2}$ will denote its L_2-gain.

Definition 2.2 (Desoer and Vidyasagar [3]). An operator H, defined from $L_2^{m,e}$ into $L_2^{p,e}$, has *a finite incremental gain* if there exists a finite non-negative constant η such that $\|Hu_1 - Hu_2\|_2 \leqslant \eta\|u_1 - u_2\|_2$ for all $u_1, u_2 \in L_2^m$. $\|H\|_\Delta$, the minimum value of η, is called the incremental gain of H.

Definition 2.3 (Desoer and Vidyasagar [3]). An operator H, defined from $L_2^{m,e}$ into $L_2^{p,e}$, is *incrementally stable* if it is stable, i.e. it maps L_2^m to L_2^p, with a finite incremental gain.

Definition 2.4 (Willems [15]). Given a causal operator H, defined from L_2^m into L_2^p, let $u_0 \in L_2^m$ and assume there exists a bounded linear operator $DH|_{u_0}$ from L_2^m into L_2^p such that

$$H(u_0 + h) = H(u_0) + DH|_{u_0}h + \alpha(h)\|h\|_2,$$

with $\lim_{\|h\|_2 \to 0}\|\alpha(h)\|_2 = 0$, then $DH|_{u_0}$ is called the *linearization* of H at u_0.

Let H be a causal operator from $L_2^{m,e}$ into $L_2^{p,e}$, then the operator $DH|_{u_0}$ from $L_2^{m,e}$ into $L_2^{p,e}$ is called the *linearization* of H at u_0 if, it is linear and if for all $T \in R^+$, $P_T DH|_{u_0}$ is a linearization of $P_T H$ at $P_T u_0$.

Let $H_{x_{0r}}$ be the input output map, defined from $L_2^{m,e}$ into $L_2^{p,e}$, associated with the state space representation

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), \\ y(t) = r(t, x(t), u(t)), \\ x(0) = x_{0r}, \end{cases} \tag{1}$$

where $u(t) \in R^m$, $y(t) \in R^p$, $x(t) \in R^n$ and the input signal u belongs to $L_2^{m,e}$. Moreover, f, r are supposed sufficiently smooth such that the system is well-defined, i.e., $\forall u \in L_2^{m,e}$, $x_{0r} \in R^n$ the solution $x(\cdot)$ is unique and $y \in L_2^{p,e}$, $\varphi : R^+ \times R^n \times L_2^{m,e} \to R^n$ denotes the state transition map of $H_{x_{0r}}$.

The unperturbed motion associated with a particular input $u_r(t)$ is denoted by $x_r(t)$; it is the solution of the differential equation (1) under the input $u_r(t)$, initialized at x_{0r} (i.e. $x_r(t) = \varphi(t, x_{0r}, u_r(t))$).

Definition 2.5 (Hahn [7] and Zubov [21]). The unperturbed motion $x_r(t)$, is said to be *uniformly asymptotically stable* in the sense of Lyapunov if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $t \geqslant 0$ and $\|x_0 - x_{0r}\| \leqslant \delta$, one has

$$\|\varphi(t, x_0, u_r(t)) - x_r(t)\| \leqslant \varepsilon$$

and

$$\lim_{t \to \infty}\|\varphi(t, x_0, u_r(t)) - x_r(t)\| = 0.$$

If these two properties hold for any $x_0 \in R^n$, the unperturbed motion is called *globally uniformly asymptotically stable*.

We recall that a function α, from R^+ to R^+, is of class K if it is continuous and strictly increasing with $\alpha(0) = 0$. Moreover if $\lim_{x \to +\infty}\alpha(x) = +\infty$ then α is of class K_∞.

Definition 2.6 (Willems [16] and Hill and Moylan [8]). A state x_0 is called *uniformly observable* if there exists a function α of class K_∞ such that for any $x_1 \in R^n - \{x_0\}$, there exists a constant $T \geqslant 0$ such that

$$\|r(t, \varphi(t, x_1, 0), 0)\|_{2,\tau}^2 \geqslant \alpha(\|x_1 - x_0\|).$$

Definition 2.7 (Willems [16] and Hill and Moylan [8]). The state space is *reachable from* x_0 if, for every $x_1 \in R^n$, there exist a finite time $T \geqslant 0$ and $u_c \in L_2^{m,e}$ such that

$$x_1 = \varphi(T, x_0, u_c).$$

Moreover, the state space is said to be *uniformly reachable from* x_0 if there exists a function α of class K such that

$$\|u_c\|_{2,T}^2 \leqslant \alpha(\|x_1 - x_0\|).$$

3. Incremental stability and stability of the motion

Before stating the main result of this note, let us associate with the input $u_r(t) \in L_2^{m,\mathrm{e}}$, the input–output map

$$y_G = G_{x_0}[u_r, y_r](\bar{u}) \triangleq H_{x_0}(u_r + \bar{u}) - y_r \tag{2}$$

where $y_r = H_{x_{0r}}(u_r)$.

In [5], the incremental gain of $G_{x_{0r}}[u_r, y_r]$ is linked to the incremental gain of $H_{x_{0r}}$. More precisely, one has

Lemma 1 (Fromion et al. [5]). *If $H_{x_{0r}}$ has finite incremental gain then, for any $u_r \in L_2^{m,\mathrm{e}}$ and $y_r \in L_2^{p,\mathrm{e}}$, $G_{x_{0r}}[u_r, y_r]$ has the same finite incremental gain.*

Proof. Recalling that [15, Theorem 2.1] the Lipschitz constant of an operator H on L_2 or on the extended space L_2^{e} are equal, one has for all $u_1, u_2 \in L_2^m$

$$\|H_{x_{0r}}(u_1) - H_{x_{0r}}(u_2)\|_2 \leqslant \eta \|u_1 - u_2\|_2$$

and for all $u_1, u_2 \in L_2^{m,\mathrm{e}}$,

$$\|H_{x_{0r}}(u_1) - H_{x_{0r}}(u_2)\|_{2,T} \leqslant \eta \|u_1 - u_2\|_{2,T}, \qquad \forall T \in R^+.$$

In our case, for any $u_r \in L_2^{m,\mathrm{e}}$ and $y_r \in L_2^{p,\mathrm{e}}$, the last inequality can be rewritten as

$$\|(H_{x_{0r}}(u_r + u_1) - y_r) - (H_{x_{0r}}(u_r + u_2) - y_r)\|_{2,T} \leqslant \eta \|u_1 - u_2\|_{2,T}$$

for all $u_1, u_2 \in L_2^m$ and all $T \in R^+$, so that $\forall u_1, u_2 \in L_2^m$,

$$\|G_{x_r}[u_r, y_r](u_1) - G_{x_r}[u_r, y_r](u_2)\|_2 \leqslant \eta \|u_1 - u_2\|_2$$

which achieves the proof of Lemma 1. □

Let us now state the main result of this note.

Theorem 1. *Let $H_{x_{0r}}$ be given with finite incremental gain. If $G_{x_{0r}}[u_r, y_r]$ has an equilibrium point uniformly observable and a state space which is uniformly reachable from it, then the unperturbed motion associated with $u_r(t)$ is uniformly globally asymptotically stable.*

Proof. The proof works out showing that a possible state space representation of $G_{x_0}[u_r, y_r]$ is the one associated with the difference between the perturbed and the unperturbed motions. Then, using Lemma 1, we prove that $G_{x_{0r}}[u_r, y_r]$ is L_2-gain stable. Classical arguments concerning the link between L_2-gain stable and Lyapunov stability are used to conclude.

Let us build a specific state space representation of $G_{x_{0r}}[u_r, y_r]$. For this, let us define the new variable

$$x(t) = x_r(t) + \bar{x}(t), \qquad u(t) = u_r(t) + \bar{u}(t),$$

so that the differential equation satisfied by $\bar{x}(t)$ is given by

$$\begin{cases} \dot{\bar{x}}(t) = F(t, \bar{x}(t), \bar{u}(t)), \\ \bar{x}(0) = x_0 - x_{0r}, \end{cases} \tag{3}$$

with

$$F(t, \bar{x}(t), \bar{u}(t)) \triangleq f(\bar{x}(t) + x_r(t), \bar{u}(t) + u_r(t)) - f(x_r(t), u_r(t)).$$

Let us now consider the following readout function:

$$r_G(t, \bar{x}(t), \bar{u}(t)) \triangleq r(\bar{x}(t) + x_r(t), \bar{u}(t) + u_r(t)) - r(x_r(t), u_r(t)), \tag{4}$$

so getting a specific state space representation of $G_{x_0}[u_r, y_r]$ because

$$y_G = r_G(t, \bar{x}(t), \bar{u}(t)).$$

On these bases, because $\bar{x} = 0$ is the equilibrium of (3) (i.e. $F(t, 0, 0) = 0$), to show the asymptotic stability of $\bar{x} = 0$ is equivalent to show the asymptotic stability of the unperturbed motion of $H_{x_{0r}}$ [8, 21].

Let us now prove that $\bar{x} = 0$ is a uniformly globally asymptotically stable equilibrium point of $G_{x_0}[u_r, y_r]$. To prove the L_2 stability of $G_{x_0}[u_r, y_r]$ we use the previous lemma which ensures that

$$\|G_{x_{0r}}[u_r, y_r](u_1) - G_{x_{0r}}[u_r, y_r](u_2)\|_2 \leqslant \eta \|u_1 - u_2\|_2$$

provided that the incremental boundedness of $H_{x_{0r}}$, i.e., $\exists \eta > 0$ such that $\|H_{x_{0r}}\|_\Delta \leqslant \eta$.

By definition, $G_{x_{0r}}[u_r, y_r]$ is unbiased, i.e., $G_{x_{0r}}[u_r, y_r](0) = 0$, thus $G_{x_{0r}}[u_r, y_r]$ is L_2 gain stable, i.e., $\|G_{x_{0r}}[u_r, y_r](\bar{u})\|_2 \leqslant \eta \|\bar{u}\|_2$.

Finally, because the assumptions ensure that $\bar{x} = 0$ is uniformly observable and that the state space is uniformly reachable from $\bar{x} = 0$, arguing as in Theorem 3 in [16], one concludes that the unperturbed motion is uniformly globally asymptotically stable. □

Corollary. *Under the assumptions of Theorem* 1, *the output is uniformly globally asymptotically stable*; *i.e. for* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that for all* $t \geqslant 0$ *and* $\|x_0 - x_{0r}\| \leqslant \delta$, *one has*

$$\|H_{x_0}(u_r(t)) - H_{x_{0r}}(u_r(t))\| \leqslant \varepsilon$$

and

$$\lim_{t \to \infty} \|H_{x_0}(u_r(t)) - H_{x_{0r}}(u_r(t))\| = 0.$$

Proof. $G_{x_{0r}}[u_r, y_r]$ is incrementally stable, thus Theorem 1 in [16] ensures that there exists a Lipschitz-continuous readout function. Simple manipulations on the norms enable to conclude. □

Remark. In fact, if $G_{x_{0r}}[u_r, y_r]$ possesses an equilibrium point uniformly observable and a state space which is uniformly reachable from this equilibrium point, the corollary ensures uniform global asymptotic stability of the output map for any dynamical system with a finite incremental gain.

Let us now discuss the reciprocity of Theorem 1. For this, we assume that f and r are uniformly Lipschitz and differentiable. Furthermore, the jacobian matrix $\partial f/\partial x$ is assumed locally Lipschitz uniformly in t. Then the following result can be proved.

Theorem 2. *If for any input*, $u_r(t) \in L_2^{m,e}$, *the associated unperturbed motion of* $H_{x_{0r}}$ *is uniformly asymptotically stable, then* $H_{x_{0r}}$ *has a finite incremental gain.*

The proof of Theorem 2 is based on the fact that the uniform asymptotic stability of the linearization is ensured by the uniform asymptotic stability of the nonlinear system.

More precisely, consider the nonlinear nonautonomous system

$$\dot{x}(t) = f(t, x(t)), \tag{5}$$

where $f: R^+ \times U_0 \to R^n$ is continuously differentiable and its derivative is bounded and Lipschitz on U_0 with $U_0 = \{x \mid \|x\| < r_0\}$.

Lemma 2. *If the equilibrium point of the nonlinear system* (5) *is uniformly asymptotically stable then the equilibrium point of*

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$$

with $A(t) = \partial f(t, 0)/\partial x$ *is uniformly asymptotically stable.*

Proof. Because of the uniform asymptotic stability of (5), there exists [10, 7, 9] a Lyapunov function $V: R^+ \times U_0 \to R$ which verifies the following properties:

(i) $\alpha_1(\|x\|) \leqslant V(t, x) \leqslant \alpha_2(\|x\|)$,
(ii) $\partial V/\partial t + (\partial V/\partial x) f(t, x) \leqslant -\alpha_3(\|x\|)$,
(iii) $\|\partial V/\partial x\| \leqslant \alpha_4(\|x\|)$,

where the α_i are class K-functions defined on U_0.

Let us rewrite f as the approximation

$$x(t) = A(t)x(t) + g(t, x(t)).$$

The mean value theorem ensures that there exists ξ belonging to a line segment connecting x to the origin such that

$$g(t, x(t)) = \left(\frac{\partial f(t, \xi)}{\partial x} - \frac{\partial f(t, 0)}{\partial x}\right) x.$$

The Lipschitz assumption on the jacobian matrix of f ensures that there exists L such that

$$\left\| \left(\frac{\partial f(t, z)}{\partial x} - \frac{\partial f(t, 0)}{\partial x}\right) x \right\| \leqslant L\|x\|\|z\|,$$

so $\forall \varepsilon$, there exists $\delta(\varepsilon)$ such that

$$\|g(t, x)\| \leqslant \varepsilon \|x\| \quad \forall x \in U_\varepsilon.$$

with $U_\varepsilon = \{x \mid \|x\| < \delta(\varepsilon)\}$.

Condition (ii) above ensures that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} A(t)x \leqslant -\alpha_3(\|x\|) - \frac{\partial V}{\partial x} g(t, x)$$

and because of (iii), one gets the majoration

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} A(t)x \leqslant -\alpha_3(\|x\|) + \alpha_4(\|\bar{x}\|)\varepsilon\|x\|.$$

Because α_3 and α_4 are class K-functions, there exist $\varepsilon_g > 0$ and a class K-function, α_5, such that $\forall x \in U_{\varepsilon_g}$

$$-\alpha_3(\|x\|) + \alpha_4(\|x\|)\varepsilon_g\|x\| \leqslant -\alpha_5(\|x\|).$$

It results that for $r_A = \min\{r_0, \delta(\varepsilon_g)\}$, one has on $U_A = \{x \mid \|x\| < r_0\}$,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} A(t)x \leqslant -\alpha_5(\|x\|)$$

and thus one concludes the uniform global asymptotic stability of the linear system. □

Remark. If we add in Theorem 1 the assumptions that f and r are uniformly Lipschitz and differentiable and that the jacobian matrix $\partial f/\partial x$ is locally Lipschitz then we can claim that the unperturbed state motion is not only globally uniformly stable but also locally exponentially stable.

Proof of Theorem 2. The assumptions ensure that $H_{x_{0r}}$ admits a linearization for any $u_r(t) \in L_2^{m,e}$, so that we can consider its linearization along $u_r(t) \in L_2^{m,e}$ and a possible state space representation is given by

$$\begin{aligned}\Delta x(t) &= \frac{\partial f}{\partial x}(t, x_r(t), u_r(t))\Delta x(t)\\ &\quad + \frac{\partial f}{\partial u}(t, x_r(t), u_r(t))\Delta u(t),\\ \Delta y(t) &= \frac{\partial r}{\partial x}(t, x_r(t), u_r(t))\Delta x(t)\\ &\quad + \frac{\partial r}{\partial u}(t, x_r(t), u_r(t))\Delta u(t), \qquad (6)\\ \Delta x(0) &= 0,\end{aligned}$$

where $x_r(t)$ is the solution of system (1) under the input $u_r(t)$ with $x(0) = x_{0r}$, $\Delta u(t) := u(t) - u_r(t)$, $\Delta x(t) := x(t) - x_r(t)$ and $\Delta y(t) := y(t) - y_r(t)$.

Now, recalling Lemma 7.1 in [15] which links the incremental gain of a nonlinear operator with the L_2-gain of its linearization according to

$$\|H_{x_{0r}}\|_\Delta = \sup_{u \in L_2^m} \|DH_{x_{0r}}|_u\|_{i_2}$$

and recalling Theorem 3.1 in [19] which claims that a uniformly asymptotically stable linear system is L_2 gain stable, we can conclude using Lemma 2. □

4. Conclusion

A link between the input–output stability and the internal stability has been pointed out in this note. The direct and reciprocal results clarify in input/output terms the requirements attached to some classical control techniques as gain scheduling control. More generally, they point out the role played by input/output continuity in a nonlinear context.

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