Asymptotic Properties of Incrementally Stable Systems

V. Fromion, S. Monaco, and D. Normand-Cyrot

Abstract—**It is shown that incremental stability of an input–output operator ensures asymptotic stability of any equilibrium pair (x_e, u_e) of its state representation if suitable minimality assumptions hold.**

I. Introduction

It is well known that under globally uniform observability and reachability conditions, L_2-gain stability implies global asymptotic stability of the equilibrium associated to the null control [17], [5]. In [5], with reference to multiple equilibria, a local result is obtained, illustrating the necessity to distinguish between L_2-gain stability (i.e., $||y||_2 \leq \gamma||u||_2$) and weak L_2-gain stability (i.e., $||y||_2 \leq \gamma||u||_2 + \beta$); the later case does not imply Lyapunov stability.

In many applications (flight dynamics, chemical processes) it is important to ensure asymptotic stability, not only at the equilibrium associated to the zero input, but also at equilibria associated to various constant controls. This problem is discussed in the present paper. It is shown that under a suitable minimality assumption, incremental stability of the input–output operator ensures asymptotic stability of any equilibrium pair (x_e, u_e).

The interest of this result is easily understood in the context of linear systems perturbed by memoryless nonlinearities, where the incremental small gain theorem can be used [13], [20], [2] or in the H_∞ control to extend linear concepts to a nonlinear domain [3], [4].

The organization of this paper is as follows. In Section II some notations and usual definitions are recalled. Then, the links between incremental stability, finite L_2-gain stability, and asymptotic stability are studied in Sections III and IV.

II. Notations

The notations and terminology recalled hereafter are classical in an input-output context [13], [20], [16], [2], [11], [7]. Denoting by E the set of real measurable n vector valued functions of the real variable t on R^+, one defines $L_2^n = \{x \in E| ||x||_2 < \infty\}$, where $||x||_2 = \sqrt{\int_0^\infty x(t)^T x(t)\, dt}$, and $L_2^{n,e} = \{x \in E | P_\tau x \in L_2^n, \forall \tau \in R^+\}$ its associated extended space, and P_τ is the causal operator which truncates a signal at time τ. For the sake of simplicity, one sets $||u||_{2,\tau} \triangleq ||P_\tau u||_2$.

The input–output map $H_{x_0}: L_2^{m,e} \rightarrow L_2^{p,e} (u \mapsto y = H_{x_0}(u))$ is assumed to be causal and invariant under the shifts. $\varphi: R^+ \times X \times L_2^{m,e} \rightarrow X$ and $r: X \times R^m \rightarrow R^p$ are the state-transition map and the output map of a time invariant state representation of H_{x_0} with state space X, a suitable subset of a normed vector space; [17] and x_0 denote the initial state. Z_e denotes the set of its equilibria, i.e., $Z_e = \{(x_e, u_e) \in X \times R^m | \varphi(t, x_e, u_e) = x_e\ \forall t \in R^+\}$.

Definition 2.1 [5]: H_{x_0} is weakly L_2-gain stable if there exist finite nonnegative constants γ and β such that $||H_{x_0}(u)||_2 \leq \gamma||u||_2 + \beta$. Its gain is the minimum value of γ. When $\beta = 0$, the system is said to be L_2-gain stable, and $||H_{x_0}||_{i_2}$ will denote its L_2-gain.

Definition 2.2: H_{x_0} has a finite incremental gain if there exists a finite nonnegative constant η such that $||H_{x_0}(u_1) - H_{x_0}(u_2)||_2 \leq \eta||u_1 - u_2||_2$ for all $u_1, u_2 \in L_2^m$. $||H_{x_0}||_\Delta$, the minimum value of η, is called the incremental gain of H_{x_0}.

Definition 2.3 [13], [20], [16]: H_{x_0} is incrementally stable if it is stable, i.e., it maps L_2^m to L_2^p and has a finite incremental gain.

We recall that a function α, from R^+ to R^+, is of class K if it is continuous and strictly increasing with $\alpha(0) = 0$. Denoted by $d(x, \Omega)$, the distance from x to Ω, i.e., $d(x, \Omega) = \inf_{x \in \Omega} ||x - x_0||_X$, some definitions concerning observability and reachability are now recalled.

Definition 2.4 [5]: A region $X_1 \subseteq X$ is zero-state detectable (ZSD) with respect to $\Omega \subseteq X$ if there exists a function α of class K and a constant $T \geq 0$ such that for any $x_1 \in X_1 - \Omega$

$$||r(\varphi(t, x_1, 0), 0)||_{2,T}^2 \geq \alpha(d(x_1, \Omega)).$$

x_0 is, moreover, uniformly observable when the set Ω reduces to a singleton $\{x_0\}, X_1 = X$ and $\lim_{x \rightarrow +\infty} \alpha(x) = +\infty$.

Definition 2.5 [5]: A region $X_1 \subseteq X$ is reachable from $\Omega \subseteq X$ if for every $x_1 \in X_1$ there exist $x_0 \in \Omega$, a finite time $T \geq 0$, and $u_c \in L_2^{m,e}$ such that $x_1 = \varphi(T, x_0, u_c)$.

Moreover, X_1 is said to be uniformly reachable from Ω if there exists a function α of class K such that

$$||u_c||_{2,T}^2 \leq \alpha(d(x_1, \Omega)).$$

If Ω reduces to a singleton $\{x_0\}$, and if $X_1 = X$, the system is said to be uniformly reachable from x_0.

III. Finite Incremental Gain and Finite Gain Stability

Given the operator H_{x_0}, let us denote by $G_{x_0}[u_e, y_e]$ the following system:

$$G_{x_0}[u_e, y_e](u) \triangleq H_{x_0}(u + u_e) - y_e$$

where $y_e = H_{x_e}(u_e)$ and clearly $G_{x_e}[u_e, y_e](0) = 0$.

It is shown in this section that under reachability of any state in Z_e, finite incremental gain of H_{x_0} implies L_2-gain stability of $G_{x_e}[u_e, y_e]$ for any particular equilibrium pair (x_e, u_e) of Z_e. More precisely, one has the following theorem.

Theorem 1: If the nonlinear operator, H_{x_0}, has finite incremental gain η, and if any state in Z_e is reachable from x_0, then for any fixed pair $(x_e, u_e) \in Z_e$ the associated system $G_{x_e}[u_e, y_e]$ is L_2-gain stable and has an L_2-gain less or equal to η.

Proof: The result is deduced from two basic lemmas proven below. First, we show in Lemma 1 that incremental boundedness is preserved when replacing the initial state x_0 by any state x_i reachable from x_0. Then, noting that u_e and y_e do not belong to L_2 but to L_2^e, we use Lemma 2, stated below in a more general context, to prove that $G_{x_0}[u_e, y_e]$ has the same incremental gain as H_{x_0}.

With this in mind, it is easy to verify that for any $(x_e, u_e) \in Z_e$, the following inequality holds:

$$||G_{x_e}[u_e, y_e](u)||_2 \leq \eta||u||_2 \quad \forall u \in L_2^m.$$

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From Lemma 1 and 2, one has

$$||G_{x_e}[u_e, y_e]||_\Delta \le \eta$$

and because of the definition of $G_{x_0}[u_e, y_e]$ and $G_{x_e}[u_e, y_e](0) = 0$

$$||G_{x_e}[u_e, y_e](u) - G_{x_e}[u_e, y_e](0)||_2 \le \eta ||u - 0||_2.$$

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Let us now state the two basic lemmas.

Lemma 1: Let $x_i \in X$, a reachable state from x_0; if H_{x_0} has a finite incremental gain, then H_{x_i} has the same finite incremental gain.

Proof of Lemma 1: The reachability assumption ensures the existence of the inputs

$$u_1(t) = \begin{cases} \hat{u}(t) & -\bar{t} \le t \le 0 \\ \tilde{u}_1(t) & t > 0 \end{cases}$$

$$u_2(t) = \begin{cases} \hat{u}(t) & -\bar{t} \le t \le 0 \\ \tilde{u}_2(t) & t > 0 \end{cases}$$

such that setting $x(-\bar{t}) = x_0$ and $x(0) = x_i$, and because of the incremental boundedness of H_{x_0}, one has

$$\begin{aligned} &||H_{x_0}(u_1) - H_{x_0}(u_2)||_2 \\ &\quad = ||H_{x_0}(\hat{u}) + H_{x_i}(\tilde{u}_1) - H_{x_0}(\hat{u}) + H_{x_i}(\tilde{u}_2)||_2 \\ &\quad \le \eta ||(\hat{u} + \tilde{u}_1) - (\hat{u} + \tilde{u}_2)||_2 \end{aligned}$$

and then $\forall \tilde{u}_1, \tilde{u}_2 \in L_2^m$

$$||H_{x_i}(\tilde{u}_1) - H_{x_i}(\tilde{u}_2)||_2 \le \eta ||\tilde{u}_1 - \tilde{u}_2||_2$$

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Before stating Lemma 2, let us define the input–output map

$$G_{x_0}[\tilde{u}, \tilde{y}](u) \triangleq H_{x_0}(\tilde{u} + u) - \tilde{y} \qquad (1)$$

where $\tilde{u}$ and $\tilde{y}$ belong to $L_2^{m,e}$ and $L_2^{p,e}$, respectively.

Lemma 2: If H_{x_0} has a finite incremental gain, then for any $\tilde{u} \in L_2^{m,e}$ and $\tilde{y} \in L_2^{p,e}, G_{x_0}[\tilde{u}, \tilde{y}]$ has the same finite incremental gain.

Proof of Lemma 2: Recalling that [16, Theorem 2.1] the Lipschitz constant of an operator H on L_2 or on the extended space L_2^e are equal, one has for all $u_1, u_2 \in L_2^m$

$$||H_{x_0}(u_1) - H_{x_0}(u_2)||_2 \le \eta ||u_1 - u_2||_2$$

and for all $u_1, u_2 \in L_2^{m,e}$

$$||H_{x_0}(u_1) - H_{x_0}(u_2)||_{2,T} \le \eta ||u_1 - u_2||_{2,T} \quad \forall T \in R^+.$$

In our case, for any $\tilde{u} \in L_2^{m,e}$ and $\tilde{y} \in L_2^{p,e}$, the last inequality can be rewritten as

$$\begin{aligned} &||(H_{x_0}(\tilde{u} + \tilde{u}_1) - \tilde{y}) - (H_{x_0}(\tilde{u} + \tilde{u}_2) - \tilde{y})||_{2,T} \\ &\quad \le \eta ||\tilde{u}_1 - \tilde{u}_2||_{2,T} \quad \forall \tilde{u}_1, \tilde{u}_2 \in L_2^m \quad \text{and} \quad \forall T \in R^+ \end{aligned}$$

so that from (1), one concludes $\forall \tilde{u}_1, \tilde{u}_2 \in L_2^m$

$$||G_{x_0}[\tilde{u}, \tilde{y}](\tilde{u}_1) - G_{x_0}[\tilde{u}, \tilde{y}](\tilde{u}_2)||_2 \le \eta ||\tilde{u}_1 - \tilde{u}_2||_2$$

which achieves the proof of Lemma 2.

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Remark: Under the assumption that H_{x_0} be a nonlinear differentiable operator [16], combining Theorem 1 and [16, Lemma 7.1] which links the incremental norm of a differentiable nonlinear operator H with the norm of its linearizations, we can deduce that for any fixed pair $(x_e, u_e) \in Z_e$, the associated linearization, $DG_{x_e}[u_e, y_e]|_0$, has L_2-gain $\gamma \le \eta$, i.e., $||DG_{x_e}[u_e, y_e]|_0||_\infty \le \eta$. This comment provides an interesting result in the context of H_∞—gain scheduling.

IV. Incremental Stability and Asymptotic Stability

On the basis of Theorem 1, two results about the internal stability of the given plant around any equilibrium in Z_e are now stated. For this purpose, we need to link the stability of H_{x_0} to the internal stability of its state-space representation. This can be done under the following assumption.

Assumption 4.1: For any pair $(x_e, u_e) \in Z_e$, let us consider $G_{x_0}[u_e, y_e]$. Its equilibrium point x_e is uniformly observable, and its state space is uniformly reachable from x_e.

The following theorem provides a global result about the stability of any equilibrium.

Theorem 2: If H_{x_0} has finite incremental gain and if Assumption 4.1 holds, then x_e is a globally asymptotically stable equilibrium of $G_{x_0}[u_e, y_e]$.

Proof: It is deduced from a classical result which links dissipative properties and Lyapunov stability [17].

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Remarks:

1) An extensive discussion on the link between input–output stability and internal stability in terms of dissipative arguments [18] can be found in [17], [6], [8], [10], [15], [14], [7], and [1]. In Assumption 4.1 uniform reachability can be replaced by some smoothness assumption of the storage function associated to (1) [1], [7].
2) The relevance of Theorem 2 lies in the study of linear systems perturbed by incrementally stable nonlinearities [20].

In presence of multiequilibria associated to a given constant control, a local result is now stated under Assumption 4.2 below.

Let $\Omega_e(u_e) = \{x_m \in X | \varphi(t, x_m, u_e) = x_m \; \forall t \in R^+\}$ and assume it to be not empty.

Assumption 4.2: There exists $d_1 > 0$ such that $X_1 = \{x \in X | d(x, \Omega_e(u_e)\} \le d_1\}$ is uniformly reachable from $\Omega_e(u_e)$ and X_1 is ZSD with respect to $\Omega_e(u_e)$.

Theorem 3: If H_{x_0} has a finite incremental gain, if any state in $\Omega_e(u_e)$ is reachable from x_0, and if Assumption 4.2 holds, then there exists $d_2 > 0$ such that under the control u_e, all the trajectories starting from $X_2 = \{x \in X | d(x, \Omega(u_e)) \le d_2\}$ remain in X_1 and asymptotically approach $\Omega_e(u_e)$.

Proof: The proof can be achieved by making use of the same arguments as in [5].

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Note that by assuming uniform observability, only global attractiveness of the equilibrium set can be obtained [17], [15].

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A Unified Approach to Stability Robustness for Uncertainty Descriptions Based on Fractional Model Representations

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Abstract—**The powerful standard representation for uncertainty descriptions in a basic perturbation model based on a standard plant representation can be used to attain necessary and sufficient conditions for stability robustness within various uncertainty descriptions. In this paper, these results are employed to formulate necessary and sufficient conditions for stability robustness of several uncertainty sets based on unstructured additive coprime factor uncertainty, gap-metric uncertainty, as well as the recently introduced Λ-gap uncertainty.**

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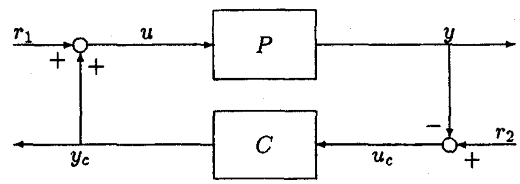

Fig. 1. Feedback connection structure $\mathcal{T}(P, C)$ of a plant P and a controller C.

I. Introduction

In a model-based control design paradigm, the design is based on a (necessarily) approximative model $\hat{P}$ of a plant to be controlled. An apparently successful control design leads to a controller C having some desired closed-loop properties for the feedback controlled model $\hat{P}$, but due to the mismatch between the actual plant P_o and the model $\hat{P}$, a verification of these desired closed-loop properties is preferred before implementing the controller C on the actual plant P_o. In this paper the discussion is directed toward the verification of one of the most important closed-loop properties: stability.

To evaluate stability when the controller C is being applied to the plant P_o, a characterization of the mismatch between the plant P_o and the model $\hat{P}$ is indispensible. Since the real plant P_o is unknown, the discrepancy in general is characterized by a so-called uncertainty set, denoted with $\mathcal{P}$. Typically, an uncertainty set $\mathcal{P}$ is defined by the (nominal) model $\hat{P}$ which is found by physical modeling or identification techniques and some bounded "area" around it [4]. The uncertainty set $\mathcal{P}$ itself reflects all possible perturbations of the (nominal) model $\hat{P}$ that may occur.

By defining the uncertainty set in such a way that at least the plant $P_o \in \mathcal{P}$, stability robustness results for the set $\mathcal{P}$ will reflect sufficient conditions under which the plant P_o will be stabilized by C; see [4] or [5]. In this perspective, special attention will be given in this paper to an uncertainty set $\mathcal{P}_{CF}$ which is characterized by additive perturbations on a coprime factor description of the nominal model $\hat{P}$. The specific application of such an uncertainty set description will be motivated by the favorable properties it has over a standard additive or multiplicative uncertainty set description.

Using the simple and powerful stability robustness results for a basic perturbation model in a standard plant configuration [4], [5], [15], several different uncertainty sets employing weighted and unstructured additive perturbations on a coprime factorization, gap-metric based uncertainty sets, and the recently introduced Λ-gap uncertainty sets will be shown to be closely related to each other. The contribution of this paper is in the unified treatment of these different uncertainty sets. While stability robustness results for uncertainty sets using additive perturbations on normalized (left) coprime factorizations [11] and gap-metric based uncertainty sets [10] have separately been derived before, this paper amplifies their relation, as well as the extension to a less conservative Λ-gap uncertainty set description [1], [2].

II. Preliminaries

Throughout this paper, the feedback configuration of a plant P and a controller C is denoted by $\mathcal{T}(P, C)$ and defined by the feedback connection structure depicted in Fig. 1.

A plant P and a controller C are assumed to be given by real rational transfer function matrices, and it will be assumed that the