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Approximated Solutions to Nonlinear Discrete-Time H_∞-Control

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***Abstract*— It is shown that there exists an analytic solution to the discrete Hamilton–Jacobi equation arising in the nonlinear discrete-time H_∞-control problem if and only if the H_∞-control problem associated to the linear approximated system is solvable. Starting from the solution of the Riccati equation associated to this linear problem, we show that the nonlinear solution can be computed at any desired degree of approximation. On this basis the control solution can be computed iteratively.**

I. Introduction

Following the current literature on nonlinear H_∞-control, the present note studies the problem of computing approximated solutions in a discrete-time context.

As is well known, in continuous-time H_∞-control [1], [11], [19], with reference to dynamics affine in the inputs, the feedback control law can be directly computed from the solution of a particular type of Hamilton–Jacobi equation. When the dynamics are generally nonlinear, the Hamilton–Jacobi and controller equations are mixed up, providing an implicit characterization of the solutions [12]. This situation always occurs in a discrete-time context [3]; even if the system's equations are affine in the inputs, the control law can only be implicitly characterized.

Based on the results in [19], regarding the analyticity of the continuous-time solutions and following an approach proposed in [14] (see also [15]), polynomial approximations are used in [12], [13], and [17] to set an iterative procedure for solving simultaneously the Hamilton–Jacobi and controller equations. The same basic idea is here developed in a discrete-time context in the full information and state information cases.

After proving the existence of a solution based on the invariance properties of an associated Hamiltonian system, a procedure for computing the solutions at any desired degree of approximation is proposed making use of some algebraic tools introduced in [16]. The present work thus proposes a discrete-time counterpart for several results of [12] and [19]. Preliminary results on this subject are in [8].

Algebraic preliminaries and the problem statement are given in Sections II and III, respectively. The existence of a solution is shown in Section IV while the computation of an approximated solution is developed in Section V. Some technical manipulations are reported in the Appendix.

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II. Algebraic Preliminaries

Consider an affine nonlinear discrete-time dynamics on $\mathbb{R}^n$, $x_{k+1} = x_k + A(x_k) + B(x_k)\mathcal{U}_k$, where $\mathcal{U}_k \in \mathbb{R}^p$. $A(x)$ and $B(x)$ are matrices of analytic functions of appropriate dimensions, with series expansions $A(x) = \Sigma_{i\geq 1} A^{[i]}(x)$ and $B(x) = \Sigma_{i\geq 0} B^{[i]}(x)$, where $(\cdot)^{[i]}(x)$ denotes a homogeneous polynomial of degree i in x or a vector of such polynomials.

Given $V(x)$ any analytic function from $\mathbb{R}^n$ to $\mathbb{R}$ and $\mathcal{U}(x)$ any analytic regular feedback on $\mathbb{R}^n$, let their expansions be

$$\mathcal{U}(x) = \sum_{i\geq 1}\mathcal{U}^{[i]}(x) \quad \text{and} \quad V(x) = \sum_{i\geq 2} V^{[i]}(x). \tag{1}$$

Setting $\mathcal{A}(x) \triangleq A(x) + B(x)\mathcal{U}(x)$, one obtains

$$(I + \mathcal{A})(x) = x + \sum_{i\geq 1}\mathcal{A}^{[i]}(x) \tag{2}$$

with

$$\mathcal{A}^{[i]}(x) = A^{[i]}(x) + \sum_{j=0}^{i-1} B^{[j]}(x)\mathcal{U}^{[i-j]}(x), \qquad i \geq 1.$$

To introduce the main algebraic tools used in this paper, let us recall the definition of the tensorial product of matrices. If X is a $n\times m$ matrix of elements (x_{ij}) and Y is a $k\times l$ matrix, then $X\otimes Y$ is a $nk \times ml$ matrix defined as

$$X\otimes Y \triangleq \begin{bmatrix} x_{11}Y & \cdots & x_{1m}Y \\ \vdots & & \vdots \\ x_{n1}Y & \cdots & x_{nm}Y\end{bmatrix}.$$

According to the usual Lie derivative definition

$$L_{\mathcal{A}}(\cdot) \triangleq \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\cdot)\mathcal{A}_i$$

where $\mathcal{A}_i$: $\mathbb{R}^n \to \mathbb{R}$ denotes the ith vector component of $\mathcal{A}$, one defines the tensor of Lie derivatives

$$L_{\mathcal{A}}^{\otimes 2}(\cdot) \triangleq \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(\cdot)\mathcal{A}_i\mathcal{A}_j$$

where $\mathcal{A}_i\mathcal{A}_j$ is actually the $((i-1)n+j)$th component of the tensorial product $\mathcal{A}\otimes\mathcal{A} = \mathcal{A}^{\otimes 2}$ (remember that $\mathcal{A}^{\otimes 2}$ is a vector of dimension n^2). Let, as in [16], the exponential Lie series $\Delta_{\mathcal{A}}$ be

$$\Delta_{\mathcal{A}} \triangleq \exp_{\otimes} L_{\mathcal{A}} \triangleq I + \sum_{i\geq 1}\frac{1}{i!}L_{\mathcal{A}}^{\otimes i}$$

where I represents the identity operator. It can be easily shown that the operator Δ is linear, namely $\Delta_{\mathcal{A}}(\Sigma_{i\geq 2} V^{[i]}) = \Sigma_{i\geq 2}\Delta_{\mathcal{A}}(V^{[i]})$. Moreover, because of the commutativity of the tensor product, one has

$$\begin{aligned}\Delta_{\Sigma_{i\geq 1}(A^{[i]}+B^{[i-1]}\mathcal{U})} &= \prod_{i\geq 1}^{\otimes} \Delta_{A^{[i]}+B^{[i-1]}\mathcal{U}} \\ &= \Delta_{A^{[1]}+B^{[0]}\mathcal{U}} \otimes \cdots \otimes \Delta_{A^{[i]}+B^{[i-1]}\mathcal{U}} \otimes \cdots. \end{aligned} \tag{3}$$

Considering now the composed analytic functions $V(I+\mathcal{A})$, $\partial V/\partial x|_{I+\mathcal{A}}\cdot B$, and $B^T(\partial^2 V/\partial x^2)|_{I+\mathcal{A}}\cdot B$, their series expansions are set in Lemma 1.

Because of the definition of Δ and expanding these composed functions in powers of $\mathcal{A}$, one easily gets the equalities

$$\begin{aligned} V(I+\mathcal{A}) &= \Delta_{\mathcal{A}}(V), \\ \left.\frac{\partial V}{\partial x}\right|_{I+\mathcal{A}} \cdot B &= \Delta_{\mathcal{A}} \otimes L_B(V) \quad \text{and} \\ line_p\left(B^T \left.\frac{\partial^2 V}{\partial x^2}\right|_{I+\mathcal{A}} \cdot B\right) &= \Delta_{\mathcal{A}} \otimes L_B^{\otimes 2}(V) \end{aligned}$$

where $line_p$ is an operator which transforms matrices into row vectors, i.e., if L is a $m \times p$ matrix whose lines are denoted by $L_i, i = 1,\cdots,m$, then $line_p(L)$ is a line matrix of dimension mp defined by $line_p(L) = [L_1 \cdots L_m]$.

Now substituting (2) and (1) to $I+\mathcal{A}$ and V, respectively, taking into account (3) and regrouping terms of the same degree, the required expansions are obtained.

Lemma 1: The following equalities are satisfied

$$\begin{aligned} V(x+\mathcal{A}(x)) &= \sum_{j\geq 2} \tilde{V}^{[j]}(x), \\ \left.\frac{\partial V}{\partial x}\right|_{x+\mathcal{A}(x)} \cdot B(x) &= \sum_{j\geq 1} W^{[j]}(x), \quad \text{and} \\ line_p\left(B^T(x) \left.\frac{\partial^2 V}{\partial x^2}\right|_{x+\mathcal{A}(x)} \cdot B(x)\right) &= \sum_{j\geq 0} Y^{[j]}(x) \end{aligned}$$

with

$$\begin{aligned} \tilde{V}^{[j]}(x) &= \sum_{i=0}^{j-2} P_i(V^{[j-i]}(x)), \\ W^{[j]}(x) &= \sum_{i=-1}^{j-2} Q_i(V^{[j-i]}(x)) \quad \text{and} \\ Y^{[j]}(x) &= \sum_{i=-2}^{j-2} T_i(V^{[j-i]}(x)). \end{aligned}$$

P_i, Q_i, and T_i are formal operators defined in the Appendix, of order i in the sense that they associate to any polynomial of degree j in x a polynomial of degree $i+j$ in x.

III. Problem Formulation

Consider the affine nonlinear system

$$\begin{aligned} x_{k+1} &= A(x_k) + B(x_k)\begin{bmatrix} w_k \\ u_k \end{bmatrix} \\ z_k &= C_1(x_k) + D(x_k)\begin{bmatrix} w_k \\ u_k \end{bmatrix} \end{aligned} \tag{4}$$

where $B(x) = [B_1(x)\ B_2(x)]$, $D(x) = [D_{11}(x)\ D_{12}(x)]$. $x \in \mathbb{R}^n$ is the state, the inputs are represented by $w \in \mathbb{R}^{m_1}$ (exogeneous input) and $u \in \mathbb{R}^{m_2}$ (control input), and the output by $z \in \mathbb{R}^{p_1}$ (tracking error). x and w are supposed to be available at every instant k (full information case).

The matrices $A(x), B_1(x), B_2(x), C_1(x), D_{11}(x)$, and $D_{12}(x)$ are analytic mappings of suitable dimensions defined in a neighborhood of $x = 0$ in $\mathbb{R}^n$. Standard assumptions are the existence of an equilibrium in $x_0 = 0$, i.e., $A(0) = 0$ and, provided a suitable change of coordinates, $C_1(0) = 0$.

For convenience, one denotes $R(x) = D^T(x)D(x) - \left[\begin{smallmatrix}\gamma^2 I & 0 \\ 0 & 0\end{smallmatrix}\right]$ and $S(x) = C_1^T(x)D(x)$, which are also analytic mappings.

The problem of disturbance attenuation via "full information" feedback can be expressed in the following terms: find a control $\overline{u} = \overline{u}(x,w)$ achieving:

1) Local asymptotic stability of the equilibrium $x_0 = 0$ and
2) Disturbance attenuation in the sense of making the L_2 gain of the closed-loop system less than or equal to a prescribed real number $\gamma > 0$, i.e., satisfying the inequality

$$\sum_{k=0}^{N} z_k^T z_k \leq \gamma^2 \sum_{k=0}^{N} w_k^T w_k \qquad \forall N \geq 0$$

for every sequence $w = (w_0, w_1, \cdots)$ such that the resulting trajectory remains in a neighborhood of $x_0 = 0$.

This problem has been recently studied in [3], [5]–[7]. In [3], a general solution is given.

Let $V(x)$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$. Let $\mathcal{U} \triangleq \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$ and define in a neighborhood of $(x,\mathcal{U}) = (0,0)$ in $\mathbb{R}^n \times \mathbb{R}^{m_1+m_2}$ a function $H(x,\mathcal{U})$ by

$$\begin{aligned} H(x,\mathcal{U}) &= V(A(x)+B(x)\mathcal{U}) - V(x) + C_1^T(x)C_1(x) \\ &\quad + 2S(x)\mathcal{U} + \mathcal{U}^T R(x)\mathcal{U}. \end{aligned}$$

The following result has been stated.

Theorem 1 [3]: Suppose there exists a C^k $(k \geq 2)$ positive definite function $V(x)$, locally defined in a neighborhood of $x = 0$ in $\mathbb{R}^n$, such that:

H1) $\overline{R}_{22} > 0$ and $\overline{R}_{11} - \overline{R}_{12}\overline{R}_{22}^{-1}\overline{R}_{21} < 0$, where the four-blocks matrix $\overline{R}$ is given by

$$\overline{R} \triangleq \frac{1}{2}\left.\frac{\partial^2 H}{\partial \mathcal{U}^2}\right|_{x=0,\mathcal{U}=0} = R(0) + \frac{1}{2}B^T(0)\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=0} B(0).$$

H2) The discrete Hamilton–Jacobi equation

$$H(x,\mathcal{U}^*(x)) = 0 \tag{5}$$

is satisfied, where $\mathcal{U}^*(x)$ (with $\mathcal{U}^*(0) = 0$), is the unique solution of the equation

$$\left.\frac{\partial H}{\partial \mathcal{U}}\right|_{x,\mathcal{U}^*(x)} = 0. \tag{6}$$

H3) The system $x_{k+1} = A(x_k) + B(x_k)\ \mathcal{U}^*(x_k)$ is locally asymptotically stable around $x = 0$. Then, the control

$$\overline{u}(x,w) = [\overline{R}_{22}^{-1}(x)\overline{R}_{21}(x) \quad I]\mathcal{U}^*(x) - \overline{R}_{22}^{-1}(x)\overline{R}_{21}(x)w \tag{7}$$

solves the problem, with

$$\overline{R}(x) = R(x) + \frac{1}{2}B^T(x)\left.\frac{\partial^2 V}{\partial x^2}\right|_{A(x)+B(x)\mathcal{U}^*(x)} B(x).$$

Setting $A^*(x) \triangleq A(x) + B(x)\mathcal{U}^*(x)$, (6) and (5) turn out to be

$$\left.\frac{\partial V}{\partial x}\right|_{A^*(x)} \cdot B(x) + 2S(x) + 2\mathcal{U}^{*T}(x)R(x) = 0, \tag{8}$$

$$\begin{aligned} &V(A^*(x)) - V(x) + C_1^T(x)C_1(x) \\ &\quad + 2S(x)\mathcal{U}^*(x) + \mathcal{U}^{*T}(x)R(x)\mathcal{U}^*(x) = 0. \end{aligned} \tag{9}$$

For reasons that will be explained in the sequel, one looks for a positive definite function $V(x)$ satisfying stronger hypotheses than those considered in Theorem 1, precisely H3′):

H3′) $x_{k+1} = A^*(x_k)$ is locally exponentially stable around $x = 0$, instead of H3. If such a function exists, we will say that the nonlinear problem is solvable. As a consequence, the problem here addressed is the following.

Problem Statement: Find necessary and sufficient conditions which guarantee the existence of an analytic positive definite function $V(x)$ satisfying H1), H2), and H3′), and set an iterative procedure for computing polynomial approximations at any desired degree of $V(x) = \Sigma_{k=2}^{\infty} V^{[k]}(x)$ and $\mathcal{U}^*(x) = \Sigma_{k=1}^{\infty} \mathcal{U}^{*[k]}(x)$.

IV. The Existence of a Solution to the Discrete Hamilton–Jacobi Equation

All the matrices describing system (4) being analytic, one can set $A(x) = \Sigma_{i\geq 1} A^{[i]}(x)$, $B(x) = \Sigma_{i\geq 0} B^{[i]}(x)$, $C_1(x) = \Sigma_{i\geq 1} C_1^{[i]}(x)$, $D(x) = \Sigma_{i\geq 0} D^{[i]}(x)$, $S(x) = \Sigma_{i\geq 1} S^{[i]}(x)$, and $R(x) = \Sigma_{i\geq 0} R^{[i]}(x)$. Furthermore, one denotes $A^{[1]}(x) = Ax$, $B^{[0]}(x) = B = [B_1 \;\; B_2]$, $C_1^{[1]}(x) = C_1 x$, $D^{[0]}(x) = D = [D_{11} \;\; D_{12}]$, $S^{[1]}(x) = x^T S = x^T C_1^T D$, and $R^{[0]}(x) = R = D^T D - \left[\begin{smallmatrix}\gamma^2 I & 0\\ 0 & 0\end{smallmatrix}\right]$.

According to these expansions, the linear approximation of system (4) is given by

$$x_{k+1} = Ax_k + B\begin{bmatrix} w_k \\ u_k\end{bmatrix}$$
$$z_k = C_1 x_k + D\begin{bmatrix} w_k \\ u_k\end{bmatrix}. \qquad (10)$$

The following assumptions on the linearized system will be recalled in the sequel:

A1) $rg(D_{12}) = m_2$.

A2) $A - BR^{-1}D^T C_1$ is nonsingular.

We will say that the linear approximated problem is solvable if the following condition holds.

There exists a (necessarily unique) matrix $X_\infty \geq 0$ satisfying the Riccati equation

$$A^T X_\infty A - X_\infty + C_1^T C_1 - F^T(R + B^T X_\infty B)F = 0 \qquad (11)$$

where $F = -(R + B^T X_\infty B)^{-1}\,(B^T X_\infty A + S^T)$ is such that $\rho(A + BF) < 1$, and there exists a nonsingular matrix $T = \left[\begin{smallmatrix}T_{11} & 0\\ T_{21} & T_{22}\end{smallmatrix}\right]$ satisfying $R + B^T X_\infty B = T^T JT$ *with* $J = \left[\begin{smallmatrix}-I & 0\\ 0 & I\end{smallmatrix}\right]$.

These conditions are necessary for the existence of the nonlinear function $V(x)$, as shown in the following proposition.

Proposition 1: Suppose there exists a C^2 positive definite function $V(x)$ satisfying H1), H2), and H3′). Then the linear approximated problem is solvable, and $2X_\infty$ is the Hessian matrix of $V(x)$ at $x = 0$.

Proof: Denoting

$$P \triangleq \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_{x=0} \geq 0 \quad \text{and} \quad F_P \triangleq \frac{\partial \mathcal{U}^*}{\partial x}\bigg|_{x=0}$$

H3′) implies that $A + BF_P$ is stable, and it is straightforward to see that under H1), the matrix

$$T = \begin{bmatrix} T_{11} & 0 \\ T_{22}\overline{R}_{22}^{-1}\overline{R}_{21} & T_{22}\end{bmatrix}$$

with T_{11} and T_{22} such that

$$-T_{11}^T T_{11} = \overline{R}_{11} - \overline{R}_{12}\overline{R}_{22}^{-1}\overline{R}_{21},$$
$$T_{22}^T T_{22} = \overline{R}_{22}$$

satisfies $R + B^T PB = T^T JT$.

Besides, differentiating (9) two times and setting $x = 0$ in the resulting equality, one has

$$A^T PA - P + C_1^T C_1 + F_P^T(R + B^T PB)F_P + F_P^T(B^T PA + S^T) + (A^T PB + S)F_P = 0. \qquad (12)$$

In the same way, the differentiation of (8) leads to

$$(A + BF_P)^T PB + S + F_P^T R = 0$$

i.e., since $R + B^T PB$ is nonsingular

$$F_P = -(R + B^T PB)^{-1}(B^T PA + S^T).$$

Combining this with (12), one recovers (11)

$$A^T PA - P - F_P^T(R + B^T PB)F_P = 0$$

and thus the linear approximated problem is solvable. A matrix satisfying such conditions being unique, one has $P = X_\infty$. ■

When $X_\infty > 0$, the solvability of the linear approximated problem is also sufficient for the existence of a solution. For, consider the Hamiltonian function $\overline{H}(x,p,\mathcal{U})$ defined in a neighborhood of $(x,p,\mathcal{U}) = (0,0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{m_1+m_2}$, by

$$\overline{H}(x,p,\mathcal{U}) = p^T(A(x) + B(x)\mathcal{U}) + C_1^T(x)C_1(x) + 2S(x)\mathcal{U} + \mathcal{U}^T R(x)\mathcal{U}$$

and let us denote

$$G(x,\mathcal{U}) \triangleq C_1^T(x)C_1(x) + 2S(x)\mathcal{U} + \mathcal{U}^T R(x)\mathcal{U}.$$

From the standard assumptions made in [3], $R(x)$ is nonsingular. Then, the related Hamiltonian system is given as usual by

$$x_{k+1} = \frac{\partial \overline{H}}{\partial p}\bigg|^T_{x_k, p_{k+1}, \overline{\mathcal{U}}^*(x_k, p_{k+1})} \qquad (13)$$

$$p_k = \frac{\partial \overline{H}}{\partial x}\bigg|^T_{x_k, p_{k+1}, \overline{\mathcal{U}}^*(x_k, p_{k+1})} \qquad (14)$$

where the control

$$\overline{\mathcal{U}}^*(x,p) = -\tfrac{1}{2}R^{-1}(x)(B^T(x)p + 2S^T(x)) \qquad (15)$$

satisfies the necessary condition of optimality, i.e.,

$$\frac{\partial \overline{H}}{\partial \mathcal{U}}\bigg|_{x,p,\overline{\mathcal{U}}^*(x,p)} = 0.$$

On these bases one can establish the following result.

Theorem 2: Under A2), if the linear approximated problem is solvable with $X_\infty > 0$, then the Hamiltonian system (13)–(14) has a locally defined analytic n-dimensional stable invariant manifold which can be described by $(x, \partial V/\partial x)$ where $V(x)$ is an analytic positive definite function satisfying H1), H2), and H3').

Sketch of the Proof: (See [9] for a detailed proof.) Using the implicit function theorem, one can argue as in [8] to prove that, under A2), there exists a function $P(x,p)$, with $P(0,0) = 0$, such that (13)–(14) can be described as

$$x_{k+1} = \frac{\partial \overline{H}}{\partial p}\bigg|^T_{x_k, P(x_k,p_k), \mathcal{U}^*(x_k, P(x_k,p_k))}$$
$$p_{k+1} = P(x_k, p_k) \qquad (16)$$

or, considering its linear approximation and using a suitable change of coordinates, as

$$\begin{bmatrix} y_{k+1} \\ q_{k+1}\end{bmatrix} = \begin{bmatrix} A + BF & 0 \\ 0 & (A + BF)^{-T}\end{bmatrix}\begin{bmatrix} y_k \\ q_k\end{bmatrix} + r\left(\begin{bmatrix} y_k \\ q_k\end{bmatrix}\right) \qquad (17)$$

where $r(\cdot)$ collects the nonlinear terms. Then, one shows that, for any $2n$-dimensional vector v, the equation

$$\theta(k,v) = U_1(k)v + \sum_{i=0}^{k-1} U_1(k-i-1)r_M(\theta(i,v)) - \sum_{i=k}^{+\infty} U_2(k-i-1)r_M(\theta(i,v))$$

where

$$U_1(k) = \begin{bmatrix} (A+BF)^k & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$U_2(k) = \begin{bmatrix} 0 & 0 \\ 0 & [(A+BF)^{-T}]^k \end{bmatrix}$$

has a solution $\theta(k, v)$ which is analytic in v and satisfies (17).

Moreover, for all $v, \theta(0, v)$ satisfies the following equation of a n-dimensional manifold S, i.e., $q - q_*(y) = 0$, where, denoting as $v_1, \cdots, v_n$ the first n components of v

$$q_*(v_1, \cdots, v_n) = \left[-\left(\sum_{i=0}^{+\infty} U_2(-i-1) r_M(\theta(i, v)) \right)_{n+j} \right]_{j=1,\cdots,n}.$$

Then one proves that a trajectory of (17) initialized in S remains in S and converges asymptotically to zero.

In the (x, p) coordinates, S is characterized by the equation $p - p_*(x) = 0$. By first showing that the function

$$V(x) \triangleq \sum_{k=0}^{+\infty} G(x_k, \mathcal{U}^*(x_k, P(x_k, p_*(x_k))))$$

where x_k is the solution of (16) with $p_k = p_*(x_k)$ and $x_0 = x$, is such that

$$\frac{\partial V}{\partial x} = p_*(x)$$

one deduces that $V(x)$ and $\overline{\mathcal{U}}^*(x) \triangleq \overline{\mathcal{U}}^*(x, P(x, \partial V/\partial x))$ satisfy (8) and (9) together with

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=0} = 2X_\infty \quad \text{and} \quad \left.\frac{\partial \overline{\mathcal{U}}^*}{\partial x}\right|_{x=0} = F.$$

■

Remark 1: The reason why H3′) is considered instead of H3) is the following. To prove the existence of a solution to the discrete Hamilton–Jacobi equation, one makes use of the fact that the Hamiltonian system (13)–(14) has a n-dimensional stable invariant manifold. This requires the existence of the stabilizing solution of the Riccati equation (11). Such a solution exists only if the symplectic matrix which describes the linear approximation at $x = 0$ of the Hamiltonian system in the form (16) (or the corresponding symplectic pair if A2) does not hold) has no eigenvalue on the unit circle (see [10] for details). In this case, the corresponding solution to the discrete Hamilton–Jacobi equation and the resulting function $\mathcal{U}^*(x)$ are such that $A + BF$ is stable and thus system $x_{k+1} = A^*(x_k)$ is exponentially stable. When the symplectic matrix has at least one eigenvalue on the unit circle, however, and as a consequence there is no solution to the linear approximated problem, there may nevertheless exist (but we do not know how to prove its existence) a solution $V(x)$ to the discrete Hamilton–Jacobi equation. As it can easily be seen, the manifold $(x, \partial V/\partial x)$ is in this case still an invariant manifold of (16) but not the stable invariant manifold.

V. How to Compute This Solution and the Associated Feedback

Rewriting (8) and (9) in terms of the operators presented in the introduction and decomposing $A^*(x)$ as $x + \overline{A}(x)$ for convenience of notations, one obtains

$$\Delta_{\overline{A}} \otimes L_B(V)(x) + 2S(x) + 2\mathcal{U}^{*T}(x)R(x) = 0, \quad (18)$$

$$\Delta_{\overline{A}}(V)(x) - V(x) + C(x) + 2S(x)\mathcal{U}^*(x) + \mathcal{U}^{*T}(x)R(x)\mathcal{U}^*(x) = 0 \quad (19)$$

where $C(x) = C_1^T(x)C_1(x)$. Expanding $C(x) = \Sigma_{i=2}^{\infty} C^{[i]}(x)$ with $C^{[i]}(x) = \Sigma_{k=1}^{i-1} C_1^{[k]T}(x)C_1^{[i-k]}(x)$, substituting this expansion as well as those of $V(x)$ and $\mathcal{U}^*(x)$ to the corresponding terms into (18) and (19) and regrouping terms of the same degree in x, it leads to

$$\sum_{i=1}^{k}(Q_{i-2}(V^{[k-i+2]}(x)) + 2\mathcal{U}^{*[i]T}(x)R^{[k-i]}(x)) + 2S^{[k]}(x) = 0, \quad (20)$$

$$\sum_{i=0}^{k-1}(P_i(V^{[k-i+1]}(x)) + 2S^{[i+1]}(x)\mathcal{U}^{*[k-i]}(x) + \sum_{j=0}^{k-i-1} \mathcal{U}^{*[i+1]T}(x)R^{[j]}(x)\mathcal{U}^{*[k-i-j]}(x)) + C^{[k+1]}(x) - V^{[k+1]}(x) = 0 \quad \forall k \geq 1. \quad (21)$$

Equations (20) and (21) provide an iterative procedure for computing the control law, as shown in the proof of the following proposition.

Proposition 2: If the linear approximated problem is solvable, then (20) and (21) are solvable $\forall k \geq 1$. Moreover, under A1), control (7) can be computed at any desired degree.

Proof: First, we define for convenience

$$\mathcal{J}_k \triangleq \{j_2, j_3, \cdots, j_k : j_2 + 2j_3 + \cdots + (k-1)j_k = k\}.$$

One immediately observes that, at the first step (for $k = 1$), (20) and (21) restore the linear case, that is

$$Q_{-1}(V^{[2]}(x)) + 2\mathcal{U}^{*[1]T}(x)R^{[0]}(x) + 2S^{[1]}(x) = 0,$$

$$P_0(V^{[2]}(x)) + 2S^{[1]}(x)\mathcal{U}^{*[1]}(x) + \mathcal{U}^{*[1]T}(x)R^{[0]}(x)\mathcal{U}^{*[1]}(x) + C^{[2]}(x) - V^{[2]}(x) = 0.$$

Computing $\mathcal{U}^{*[1]}$ from the first equation and substituting it into the second one, one obtains the Riccati equation (11), for which there exists indeed a solution. As a consequence, one has $V^{[2]}(x) = x^T X_\infty x$ and $\mathcal{U}^{*[1]}(x) = -(R^{[0]}(x))^{-1}(B^T X_\infty A + S^{[1]T}(x)) = Fx$.

For $k = 2$ in (20) and (21) one may conclude that the computation of $V^{[3]}$ will require the knowledge of $\mathcal{U}^{*[2]}$ and the computation of $\mathcal{U}^{*[2]}$ the knowledge of $V^{[3]}$. After a precise analysis, however, one verifies that the coefficient of $\mathcal{U}^{*[2]}$ in (21) for $k = 2$ is equal to $2(B^T X_\infty(A + BF) + RF + S)x$ and thus is equal to zero. It results that $V^{[3]}$ can be computed from $V^{[2]}$ and $\mathcal{U}^{*[1]}$ according to (21) and then $\mathcal{U}^{*[2]}$ from $V^{[2]}$ and $\mathcal{U}^{*[1]}$ according to (20).

The same arguments work for $k > 2$. In fact, standard manipulations of the operators Q_i and R_i in (20) lead, at the generic order k, to

$$M^{[k]}(x) + 2\mathcal{U}^{*[k]T}(x)(R + B^T X_\infty B) = 0$$

where $M^{[k]}(x)$ depends on $\mathcal{U}^{*[1]}(x), \cdots, \mathcal{U}^{*[k-1]}(x)$ and $V^{[2]}(x), \cdots, V^{[k+1]}(x)$ only. More precisely, one has

$$M^{[k]}(x) = M_1^{[k]}(x) + M_2^{[k]}(x) + M_3^{[k]}(x)$$

where

$$M_1^{[k]}(x) \triangleq \sum_{i=1}^{k-1}(Q_{i-2}(V^{[k-i+2]}(x)) + 2\mathcal{U}^{*[i]T}(x)R^{[k-i]}(x)),$$

$$M_2^{[k]}(x) \triangleq \sum_{i=0}^{k-2} P_i \otimes L_{B^{[k-i-1]}}(V^{[2]}(x)) + 2S^{[k]}(x)$$

and

$$M_3^{[k]}(x) \triangleq \Delta_{\overline{A}^{[1]}} \otimes \left(\sum_{\mathcal{J}_{k-1}} \frac{1}{j_2! \cdots j_{k-1}!} L_{\overline{A}^{[2]}}^{\otimes j_2} \otimes \cdots \otimes L_{\overline{A}^{[k-1]}}^{\otimes j_{k-1}} + L_{\tilde{A}^{*[k]}} \right) \otimes L_{B^{[0]}}(V^{[2]}(x))$$

with $\tilde{A}^{[k]}(x) \triangleq A^{[k]}(x) + \Sigma_{i=1}^{k-1} B^{[k-i]}(x)\mathcal{U}^{*[i]}(x)$.

Since $R + B^T X_\infty B$ is invertible, one obtains

$$\mathcal{U}^{*[k]}(x) = -\tfrac{1}{2}(R + B^T X_\infty B)^{-1} M^{[k]T}(x) \tag{22}$$

which shows that (20) can be used to compute iteratively $\mathcal{U}^*(x)$.

As far as the computation of $V^{[k+1]}(x)$, appearing in $\mathcal{U}^{*[k]}(x)$, is concerned, analogous calculations show that it can be computed directly from (21) since the coefficient of $\mathcal{U}^{*[k]}$ in (21) is equal to $2(B^T X_\infty(A + BF) + RF + S)x = 0$ because of the expression of $\mathcal{U}^{*[1]}(x) = Fx$.

Manipulations on (21) give

$$V^{[k+1]}((A + BF)x) - V^{[k+1]}(x) + N^{[k+1]}(x) = 0 \tag{23}$$

where $N^{[k+1]}(x)$ depends on $V^{[2]}(x), \cdots, V^{[k]}(x)$ and $\mathcal{U}^{*[1]}(x), \cdots, \mathcal{U}^{*[k-1]}(x)$ only. More precisely, one has

$$N^{[k+1]}(x) = N_1^{[k+1]}(x) + N_2^{[k+1]}(x) + N_3^{[k+1]}(x)$$

where

$$N_1^{[k+1]}(x) \triangleq \sum_{i=1}^{k-2} \left(P_i(V^{[k-i+1]}(x)) + 2S^{[i+1]}(x)\mathcal{U}^{*[k-i]}(x) + \sum_{j=0}^{k-i-1} \mathcal{U}^{*[i+1]T}(x) R^{[j]}(x) \mathcal{U}^{*[k-i-j]}(x) \right)$$

$$N_2^{[k+1]}(x) \triangleq 2S^{[k]}(x)\mathcal{U}^{*[1]}(x) + C^{[k+1]}(x) + \sum_{j=1}^{k-1} \mathcal{U}^{*[1]T}(x) R^{[j]}(x) \mathcal{U}^{*[k-j]}(x)$$

and

$$N_3^{[k+1]}(x) \triangleq \Delta_{\overline{A}^{[1]}} \otimes \left(\sum_{\mathcal{J}_{k-1}} \frac{1}{j_2! \cdots j_{k-1}!} L_{\overline{A}^{[2]}}^{\otimes j_2} \otimes \cdots \otimes L_{\overline{A}^{[k-1]}}^{\otimes j_{k-1}} + L_{\tilde{A}^{[k]}} \right) (V^{[2]}(x)).$$

According to this, one can compute recursively $V^{[k+1]}(x)$ from (23) and then $U^{*[k]}(x)\ \forall k \geq 1$ from (22) using the sequence

$$U^{*[1]}(x), V^{[2]}(x), V^{[3]}(x), U^{*[2]}(x), \cdots, V^{[k]}(x), U^{*[k-1]}(x)$$

which is computed at the previous steps.

The remaining problem is to discuss the solvability of (23).

Reminding that any homogeneous polynomial $P(x)$, of degree i in x, can be written as $P(x) = P_i \cdot x^{\otimes i}$ where P_i is a line vector of dimension n^i, one sets $V^{[k+1]}(x) = V_{k+1} \cdot x^{\otimes k+1}$ and $N^{[k+1]}(x) = N_{k+1} \cdot x^{\otimes k+1}$. Equation (23) becomes

$$V_{k+1}((A + BF)^{\otimes k+1} - I^{\otimes k+1}) = -N_{k+1}.$$

By definition $I^{\otimes k+1} = I_{n^{k+1}, n^{k+1}}$, and the assumption $\rho(A + BF) < 1$ implies $\rho((A + BF)^{\otimes k+1}) < 1$ so that $(A + BF)^{\otimes k+1} - I^{\otimes k+1}$ is invertible (putting $A + BF$ in its Jordan form one can immediately see that every eigenvalue of $(A+BF)^{\otimes k+1}$ is a product of eigenvalues of $A + BF$). As a consequence

$$V^{[k+1]}(x) = N_{k+1}(I^{\otimes k+1} - (A + BF)^{\otimes k+1})^{-1} \cdot x^{\otimes k+1}. \tag{24}$$

From these formulas one can deduce an approximation of the control solution. Indeed, considering, instead of (7), the equation

$$\overline{R}_{22}(x)\overline{u}(x, w) = [\overline{R}_{21}(x) \quad \overline{R}_{22}(x)]\mathcal{U}^*(x) - \overline{R}_{21}(x)w \tag{25}$$

and rewriting $\overline{u}(x, w) = \overline{u}_1(x) + \overline{u}_2(x)w$, to solve (25) is equivalent to solve

$$\overline{R}_{22}(x)\overline{u}_1(x) = [\overline{R}_{21}(x) \quad \overline{R}_{22}(x)]\mathcal{U}^*(x)$$

and

$$\overline{R}_{22}(x)\overline{u}_2(x) = -\overline{R}_{21}(x).$$

Since $\overline{R}_{22} = D_{12}^T D_{12} + B_2^T X_\infty B_2$ is invertible, it is easy to see that, after expanding the different entities and regrouping the terms of the same order, one has

$$\overline{u}_1^{[k]}(x) = \overline{R}_{22}^{-1} \left(\sum_{i=1}^{k} [\overline{R}_{21}^{[k-i]}(x) \quad \overline{R}_{22}^{[k-i]}(x)] \mathcal{U}^{*[i]}(x) - \sum_{i=1}^{k-1} \overline{R}_{22}^{[k-i]}(x) \overline{u}_1^{[i]}(x) \right), \tag{26}$$

$$\overline{u}_2^{[k]}(x) = \overline{R}_{22}^{-1} \left(\overline{R}_{21}^{[k]}(x) + \sum_{i=0}^{k-1} \overline{R}_{22}^{[k-i]}(x) \overline{u}_2^{[i]}(x) \right) \tag{27}$$

with

$$\overline{R}_{22}^{[k]}(x) = \sum_{i=0}^{k} D_{12}^{[k-i]T}(x) D_{12}^{[i]}(x) + line_{m_2}^{-1} \left(\sum_{i=-2}^{k-2} T_i^{22}(V^{[k-i]}(x)) \right),$$

$$\overline{R}_{21}^{[k]}(x) = \sum_{i=0}^{k} D_{12}^{[k-i]T}(x) D_{11}^{[i]}(x) + line_{m_1}^{-1} \left(\sum_{i=-2}^{k-2} T_i^{21}(V^{[k-i]}(x)) \right)$$

where T_i^{22} and T_i^{21} are operators defined in the Appendix.

One thus can recursively compute $\overline{u}^{[k]}$ starting with

$$\overline{u}_1^{[1]}(x) = [\overline{R}_{22}^{-1}\overline{R}_{21} \quad I]Fx \quad \text{and} \quad \overline{u}_2^{[0]}(x) = -\overline{R}_{22}^{-1}\overline{R}_{21}$$

which provide the linear control solution. ■

Remark 2: If only state information is available, it can be proved ([3]) that assuming

H1') $\overline{R}_{22} > 0$ and $\overline{R}_{11} < 0$.

instead of H1), the control $\overline{u}(x) = u^*(x)$ solves the problem. So, the control solution is directly given at any desired order by (26)–(27).

Let us now announce a result which summarizes the entire discussion.

Theorem 3: Given system (4) with $A(x), B(x), C_1(x)$, and $D(x)$ analytic, suppose its linear approximation at $x = 0$ satisfies A1) and A2). If the linear approximated problem is solvable with $X_\infty > 0$, there exists an analytic control which provides a solution to the nonlinear H_∞ control problem for system (4). A polynomial approximation of this control, using (26) and (27), can be computed at any desired degree on the basis of the iterative computation of $V(x)$ and $\mathcal{U}^*(x)$ from (24), (22), and $V^{[2]}(x) = x^T X_\infty x, \mathcal{U}^{*[1]}(x) = Fx$.

The proof follows directly from Theorems 1 and 2 and Proposition 2.

VI. Conclusions

As is well known, a difficulty which arises when computing control laws in the discrete-time context stands in the necessity of manipulating compositions of functions. In the H_∞-control problem, this appears in the complexity for obtaining saddle-point solutions and expressing an associated Hamilton–Jacobi type equation. The contribution of the present paper is twofold. First, we prove that there exists a solution to the problem if the linear approximated problem is solvable, by explaining its connection to a standard Hamiltonian system, and then we propose a method for its iterative computation. The introduction of formal operators for manipulating compositions of functions is quite cumbersome but enables us to deal, exactly in the same way, with dynamics which are nonlinear in the input variables. Indeed, considering a nonlinear dynamics $x_{k+1} = x_k + \mathcal{A}(x_k, \mathcal{U}_k)$, one can obtain, as in Lemma 1, polynomial expansions of $V(I + \mathcal{A})$, $\partial V/\partial x|_{I+\mathcal{A}} \cdot \partial\mathcal{A}/\partial\mathcal{U}$ and $(\partial\mathcal{A}/\partial\mathcal{U})^T \cdot \partial^2 V/\partial x^2|_{I+\mathcal{A}} \cdot \partial\mathcal{A}/\partial\mathcal{U}$.

Appendix

The operators P_i, Q_i, and T_i are given by

$$P_i = \Delta_{\mathcal{A}[1]} \otimes \tilde{P}_i, \qquad \forall i \geq 0$$

$$Q_i = \sum_{j=0}^{i+1} P_j \otimes L_{B[i-j+1]} \qquad \forall i \geq -1,$$

$$T_i = \sum_{j=-1}^{i+1} Q_j \otimes L_{B[i-j+1]} \qquad \forall i \geq -2$$

with

$$\tilde{P}_i \triangleq \Sigma_{j_2,\cdots,j_{i+1}}^{j_2+2j_3+\cdots+ij_{i+1}=i} \frac{1}{j_2!\cdots j_{i+1}!} L_{\mathcal{A}[2]}^{\otimes j_2} \otimes \cdots \otimes L_{\mathcal{A}[i+1]}^{\otimes j_{i+1}}.$$

Remark 3: The formula in Lemma 1

$$line_p\left(B^T(x)\frac{\partial^2 V}{\partial x^2}\bigg|_{x+\mathcal{A}(x)} \cdot B(x)\right) = \sum_{j\geq 0} Y^{[j]}(x)$$

is actually used for matrices $B_1(x), B_2(x)$ of dimension $n \times m_1, n \times m_2$, respectively. More precisely

$$line_{m_2}\left(B_2^T(x)\frac{\partial^2 V}{\partial x^2}\bigg|_{x+\mathcal{A}(x)} \cdot B_2(x)\right) = \sum_{j\geq 0}\sum_{i=-2}^{j-2} T_i^{22}(V^{[j-i]}(x)),$$

$$line_{m_1}\left(B_2^T(x)\frac{\partial^2 V}{\partial x^2}\bigg|_{x+\mathcal{A}(x)} \cdot B_1(x)\right) = \sum_{j\geq 0}\sum_{i=-2}^{j-2} T_i^{21}(V^{[j-i]}(x))$$

where

$$T_i^{22} = \sum_{j=-1}^{i+1} Q_j^{22} \otimes L_{B_2}^{[i-j+1]}, \quad Q_i^{22} = \sum_{j=0}^{i+1} P_j \otimes L_{B_2}^{[i-j+1]},$$

$$T_i^{21} = \sum_{j=-1}^{i+1} Q_j^{22} \otimes L_{B_1}^{[i-j+1]}.$$

Remark 4: In Lemma 1, the operators P_i, Q_i, and T_i are defined according to $\mathcal{A}(x)$. In Section III, they are to be defined according to $\overline{A}(x) = A^*(x) - x$, i.e., $\mathcal{A}(x)$ must be replaced by $\overline{A}(x)$ in the formulas here given.

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