STOCHASTIC CYCLES IN VAR PROCESSES

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ABSTRACT. This paper presents an additive decomposition of the moving average representation of VAR(MA) processes into cyclical components. Each cyclical component is associated with a root of the characteristic VAR polynomial and with a given frequency. The proposed representation is unique and it provides explicit formulae for the the (dynamic) loadings of the variables onto the different cyclical components. The paper discusses relations with existing definitions of stochastic cycles. Applications of this decomposition include:

i) the computation of the fraction of each observation accounted by the different cyclical components, similarly to the permanent-transitory decomposition of Beveridge and Nelson;

ii) the definition of co-cyclicality, in the sense of common features literature, when the relevant feature consists of a cycle with given frequency;

iii) indications on how to orthogonalize the innovations in Structural VAR exercises, in order e.g. to interpret one of the shocks as the one affecting the business cycle.

The implementation of some of the above techniques on the implied reduced VARMA form of a given DSGE model and on empirical VARs would allow to ascertain if the given DSGE model can replicate stylized cyclical features found in the data.

1. Introduction

Business cycles, defined as 'economy-wide fluctuations in production or economic activity over several months or years'[1] have been widely studied in economics, see the early work of Burns and Mitchell (1946) and Stock and Watson (1999) for a more recent account. Business cycles are hence by definition a) irregular, i.e. stochastic in nature, b) common to many macroeconomic indicators, and c) characterized by a certain average periodicity.

One popular stochastic process used to represent cycles is the univariate AutoRegressive process of order 2, AR(2), usually with two complex AR roots. This is because each of these processes can be associated with a frequency, λ_u say, or, equivalently with a periodicity of $2\pi/\lambda_u$. When one lets λ_u tend to 0 (respectively π), an AR(2) with complex roots converges to an AR(1) process has 2 real positive (respectively negative) roots; usually also these processes are included in the definition of a stochastic cycle, see e.g. Harvey and Trimbur (2003). This is the class of processes that are assumed to represent cycles in this paper.

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[1]Definition taken from http://en.wikipedia.org/wiki/Business_cycle

Business cycles are by definition common to many economic time series. The loadings of multivariate systems onto univariate AR(2) cycles are explicitly built in the specification of Structural Time Series (STS) models, see e.g. Harvey (1990). Several of these models have a vector autoregressive moving average (VARMA) reduced form; the VARMA parameters are a function of the common structural cycles. Because the structural parameters are estimated directly, there is no need to recover the structural parameters of the cycle from its VARMA reduced form.

VAR(MA) models are very much in use in applied macroeconomics; they allow to discuss common trends and cycles, and they provide impulse responses to shocks. When considering a generic VARMA model, it is not straightforward to recover the common cycles hidden in it. The purpose of this paper is to provide a a representation theory which discusses the association of VARMA coefficients with (common) cycles.

Beveridge and Nelson (1981) proposed a univariate decomposition of economic time series into a random walk component (trend) and a stationary component (cycle), the so-called BN decomposition. Their results have been extended to the multivariate case, where the trend component can be common (in presence of cointegration), see e.g. Proietti (1997).

This paper proceeds in this tradition to propose a multivariate decomposition of the stationary component of a VAR(MA) into cycles, each one characterized by different frequency (and amplitude). This decomposition can be coupled with the multivariate BN decomposition to produce a joint decomposition into trends, cycles (with different frequencies), seasonals (identified with cycles with seasonal frequency) and an irregular component, which is given by a finite order MA process.

Several economic theories predict common cyclical components. For instance, the permanent income hypothesis (PIH) implies a common cycle between aggregate consumption and aggregate income, see Campbell and Mankiw (1990). This in turn implies that there exists a linear combination of consumption and income that does not include a cycles, i.e. that this cyclical feature is a common feature in the sense of Engle and Kozicki (1993). Common cycles have been widely discussed in the literature, see Vahid and Engle (1993, 1997) and Hecq, Palm, and Urbain (2006) inter alia. The decomposition presented in this paper can be used to discuss co-cyclicality as rank reduction of the loading matrices.

The present results generate tools for a descriptive decomposition of observed economic time series. These methods can also be used as model evaluation tools; for instance the reduced VARMA form of a DSGE may be scrutinized for common cycles as a stylized fact that these models are capable of replicating. A similar analysis performed on estimated VARs may reveal is these features are found in the data as well. Finally, these results may serve as a basis for tests of co-cyclicality as implied e.g. by PIH theories.

A different application of the present results is to devising new methods to orthogonalize innovations in Structural VAR exercises, in order e.g. to interpret one of the shocks as the one affecting the business cycle.

The rest of the paper is organized as follows: Section 2 introduces notation and definitions of structures of interest, Section 3 present the additive cycle decomposition, Section 4 discusses its relation with the n-order stochastic cycle of Harvey and Trimbur (2003). Section 5 describes the spectral properties of the stochastic cycles, Section 6 discusses how to obtain cancelations of the cyclical components by linear combinations and/or by filtering of the observed series. Section 7 presents examples, while Section 8 reports conclusions. In the

Appendices we present some mathematical results that are needed for the derivation of the representation results.

A final word on notation. In the following, $a := b$ and $b =: a$ indicate that a is defined by b; any sum $\sum_{n=a}^{b} \cdot$ where $b < a$ is defined equal to 0. For any matrix polynomial $\pi(z) := \sum_{n=0}^{d_\pi} \pi_n z^n$, $z \in \mathbb{C}$, $\pi_n \in \mathbb{C}^{p\times r}$ where $0 < r \leq p$, we indicate its degree by d_π, i.e. $d_\pi := \deg \pi(z)$ and $0 < d_\pi < \infty$; when $\pi_n \in \mathbb{R}^{p\times r}$ we say that $\pi(z)$ has real coefficients. For $z_u \in \mathbb{C}$, $|z_u|$ indicates its modulus.

For any full column rank matrix $\gamma \in \mathbb{C}^{p\times r}$, γ^* indicates its complex conjugate and γ' its conjugate transpose; in case γ is real, γ' reduces to the transpose. We indicate by $\operatorname{col}(\gamma)$ the linear span of the columns of γ with coefficients in the field $\mathbb{C}$ or $\mathbb{R}$ if γ is complex or real, respectively. $\gamma_\perp$ indicates a basis of $\operatorname{col}^\perp(\gamma)$, the orthogonal complement of $\operatorname{col}(\gamma)$. $\bar{\gamma} := \gamma(\gamma'\gamma)^{-1}$ so that $P_\gamma := \bar{\gamma}\gamma' = \gamma\bar{\gamma}'$ denotes the orthogonal projector matrix onto $\operatorname{col}(\gamma)$ and $M_\gamma := I - P_\gamma$ the orthogonal projector matrix onto $\operatorname{col}^\perp(\gamma)$. For a matrix A we often employ a rank factorization of the type $A = -\alpha\beta'$ where α and β are bases of $\operatorname{col}(A)$ and $\operatorname{col}(A')$, and the negative sign is chosen for convenience in the calculations. Finally, $1_{j,k}$ is the indicator function equal to 1 if $j = k$ and 0 otherwise.

2. Setup and definitions

In this section we introduce notation and state the autoregressive (AR) and moving average (MA) representation of a VAR system. We consider the vector autoregressive process (VAR) of finite order d_Π

$$\sum_{n=0}^{d_\Pi} \Pi_n X_{t-n} = \epsilon_t \tag{2.1}$$

where $\Pi_n \in \mathbb{R}^{p\times p}$, $\Pi_0 = I$ and ϵ_t is a p-dimensional martingale difference sequence (with respect to the natural filtration generated by X_t) with positive definite conditional covariance matrix Ω. A leading example of this is when ϵ_t are Gaussian i.i.d. random vectors. Deterministic components D_t are omitted from (2.1) for ease of exposition; they could be included by replacing X_t with $X_t - D_t$ or by replacing ϵ_t with $\epsilon_t + D_t$.

Indicate the AR polynomial in (2.1) by $\Pi(z) := \sum_{n=0}^{d_\Pi} \Pi_n z^n$, $z \in \mathbb{C}$, by $\det \Pi(z)$ and $\operatorname{adj}\Pi(z)$ its characteristic and adjoint polynomials. Remark that, because $\Pi(z)$ has real coefficients, so do $\det \Pi(z)$, $\operatorname{adj}\Pi(z)$ and $\operatorname{inv}\Pi(z) = \operatorname{adj}\Pi(z)/\det\Pi(z)$. It is useful to factorize the characteristic polynomials in terms of its roots; because $\Pi(0) = I$, one can write $\det\Pi(z) = \prod_{u=1}^{q}(1 - w_u z)^{a_u}$, where $w_u := z_u^{-1}$ and z_u is a root of $\det\Pi(z)$ with multiplicity $a_u > 0$. We also define $z_{\min} := \min_u |z_u|$ and observe that $z_{\min} > 0$.

The power series representation of $\operatorname{inv}\Pi(z)$ has real coefficients and it is written as

$$C(z) := \operatorname{inv}\Pi(z) = \sum_{n=0}^{\infty} C_n z^n, \quad |z| < z_{\min},$$

where $C(0) = C_0 = \operatorname{inv}\Pi_0 = \operatorname{inv} I = I$. It is well known (see e.g. Brockwell and Davis, 1987, page 408) that if $\Pi(z)$ has stable roots $z_{\min} > 1$, so that $C(z)$ is holomorphic on a disk larger than the unit disk, then the following moving average (MA) form corresponds to a linear process with second moments,

$$X_t = \sum_{n=0}^{\infty} C_n \epsilon_{t-n}, \quad C_0 = I. \tag{2.2}$$

3. Moving average decomposition

Some of the factors in $\det \Pi(z)$ could be common to $\operatorname{adj} \Pi(z)$; we state the cancelation of their common factors as the following lemma, for ease of later reference. The same lemma gives also the order of the pole of $\operatorname{inv} \Pi(z)$ at $z = z_u$, labeled m_u.

Lemma 3.1 (inv $\Pi(z)$ has pole of order m_u at $z = z_u$)**.** *One has*

$$\operatorname{adj} \Pi(z) =: G(z) \prod_{u=1}^{q} (1 - w_u z)^{b_u}, \quad 0 \leq b_u < a_u, \quad G(z_u) \neq 0,$$

where $G(z)$ has real coefficients,

$$\operatorname{inv} \Pi(z) = \frac{G(z)}{g(z)}, \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_q\},$$

where

$$g(z) := \prod_{u=1}^{q} (1 - w_u z)^{m_u}, \quad m_u := a_u - b_u > 0,$$

has real coefficients, and $\operatorname{inv} \Pi(z)$ *has a pole of order m_u at $z = z_u$.*

Note that when there are not such common factors, $G(z) = \operatorname{adj} \Pi(z)$ and $g(z) = \det \Pi(z)$. In Theorem 3.3 below, we introduce a novel representation, which we call the additive cycles (MAD) representation of X_t. This is derived from the Laurent series representation of $C(z)$ and gives an additive decomposition of the MA form $X_t = \sum_{n=0}^{\infty} C_n \epsilon_{t-n}$, where the contribution of each root to the dynamics of the process is made explicit. In (3.2) below, X_t is written as the sum of matrix polynomials $A_u(L)$, $B_u(L)$ which respectively load the MA(∞) processes $c_u(L)\epsilon_t$, $d_u(L)\epsilon_t$ plus an additional finite MA part $R(L)\epsilon_t$, which is present only when $d_G \geq d_g$, where $d_G := \deg G(z)$ and $d_g := \deg g(z)$. All the coefficients of such a representation are real and uniquely determined by $\Pi(z)$.

In order to compute it, we employ Lemma 3.2 below. This lemma shows how to construct a polynomial $B(z)$ that approximates a polynomial $P(z)$ around q points up to a given order of derivative[2], here indicated with $f^{(n)}(z_u) := \frac{\partial^n}{\partial z^n} f(z)\big|_{z=z_u}$.

Lemma 3.2. *Let $P(z)$, $z \in \mathbb{C}$, be a matrix polynomial of degree d_P and $p(z) := \prod_{u=1}^{q} (z - z_u)^{\ell_u}$ be a scalar polynomial of degree d_p, where $z_u \in \mathbb{C}$ are distinct and $\ell_u \geq 1$; then one has*

$$P(z) = B(z) + p(z)R(z),$$

where $B(z)$ is a matrix polynomial of degree $d_B = d_p - 1$ such that for $n = 0, \ldots, \ell_u - 1$ and $u = 1, \ldots, q$, one has $B^{(n)}(z_u) = P^{(n)}(z_u)$ and $R(z)$ is a matrix polynomial of degree $d_R = d_P - d_p$. In particular,

$$B(z) := \sum_{u=1}^{q} p_u(z) B_u(z), \qquad p_u(z) := \frac{p(z)}{(z - z_u)^{\ell_u}}, \qquad B_u(z) := \sum_{n=0}^{\ell_u - 1} B_{u,n} (z - z_u)^n,$$

where

$$B_{u,n} := \frac{H_u^{(n)}(z_u)}{n!}, \qquad H_u(z) := \frac{P(z)}{p_u(z)}.$$

[2] A particular case of the same formula is used to find the error correction representation of a VAR in Johansen (2009).

We apply Lemma 3.2 to $P(z) := G(z)$ and $p(z) := hg(z) = \prod_{u=1}^{q}(z - z_u)^{m_u}$, where $h := \prod_{u=1}^{q}(-z_u)^{m_u} =: (-z_u)^{m_u} h_u$. This gives

$$G(z) = \sum_{u=1}^{q} h_u g_u(z) B_u(z) + hg(z)R(z),$$

where $g(z) =: (1 - w_u z)^{m_u} g_u(z)$; dividing both by sides by $g(z)$, one has

$$\operatorname{inv}\Pi(z) = \sum_{u=1}^{q} \frac{h_u B_u(z)}{(1 - w_u z)^{m_u}} + hR(z), \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_q\}. \tag{3.1}$$

Because $\Pi(z)$ has real coefficients the complex roots come in conjugate pairs; next we group them together, and in (3.2) below we give a decomposition of X_t which exclusively involves terms with real coefficients, see Lemma A.1. We represent the reciprocal of both real and complex roots in polar form, i.e. we define (λ_u, ρ_u) from $w_u =: \rho_u e^{\mathrm{i}\lambda_u}$ with $0 \leq \lambda_u < 2\pi$, and order them using a lexicographic order[3] on the pairs (λ_u, ρ_u).

Theorem 3.3 (Moving average decomposition (MAD))**.** *Let $w_u =: \rho_u e^{\mathrm{i}\lambda_u}$ with $0 \leq \lambda_u < 2\pi$ and $h := \prod_{u=1}^{q}(-z_u)^{m_u} =: (-z_u)^{m_u} h_u$; then one has*

$$X_t = \sum_{u\,:\,0<\lambda_u<\pi} A_u(L)c_u(L)\epsilon_t + \sum_{u\,:\,\lambda_u\in\{0,\pi\}} h_u B_u(L) d_u(L)\epsilon_t + hR(L)\epsilon_t, \tag{3.2}$$

where, see Lemma 3.2, $B_u(z)$, $R(z)$ are matrix polynomials with real coefficients and degree $d_{B_u} = m_u - 1$, $d_R = d_G - d_g$,

$$A_u(z) := (1 - w_u^* z)^{m_u} h_u B_u(z) + (1 - w_u z)^{m_u} h_u^* B_u^*(z)$$

has real coefficients and degree $d_{A_u} = 2m_u - 1$, and

$$c_u(z) := \left(\sum_{n=0}^{\infty} \frac{\sin(n+1)\lambda_u}{\sin\lambda_u} \rho_u^n z^n\right)^{m_u}, \qquad d_u(z) := \left(\sum_{n=0}^{\infty} \rho_u^n z^n\right)^{m_u} \tag{3.3}$$

converge for $|z| < z_{\min}$.

The following remarks are in order:

i) Eq. (3.2) gives an additive decomposition of the MA form $X_t = \sum_{n=0}^{\infty} C_n \epsilon_{t-n}$, where the contribution of each root to the dynamics of the process is made explicit. All the coefficients in (3.2) are real and uniquely determined by $\Pi(z)$.

ii) In the first term one finds the contribution of the complex roots. In particular, the component of the dynamics of X_t which is due to the complex pair z_u, z_u^* is given by $A_u(L)c_u(L)\epsilon_t$, where $y_{u,t} := c_u(L)\zeta_t$, with ζ_t a univariate white noise, is the MA representation of the univariate $\mathrm{AR}(2)^{m_u}$ process

$$(1 - 2\rho_u \cos\lambda_u L + \rho_u^2 L^2)^{m_u} y_{u,t} = \zeta_t$$

and

$$1 - 2\rho_u \cos\lambda_u z + \rho_u^2 z^2 = (1 - w_u z)(1 - w_u^* z),$$

using $w_u =: \rho_u e^{\mathrm{i}\lambda_u} = \rho_u(\cos\lambda_u + \mathrm{i}\sin\lambda_u)$.

[3] This means $t < s$ if and only if $\lambda_t < \lambda_s$ or $(\lambda_t = \lambda_s$ and $\rho_t < \rho_s)$.

iii) When $m_u = 1$, $c_u(L)\zeta_t$ describes a cycle with period $2\pi/\lambda_u$ and amplitude ρ_u^2. The coefficients of $c_u(z)$ are $\varphi_n := \frac{\sin(n+1)\lambda_u}{\sin\lambda_u}\rho_u^n$, composed of a cyclical function $\frac{\sin(n+1)\lambda_u}{\sin\lambda_u}$ times a damping factor ρ_u^n. If $m_u = 2$, $c_u(z)$ has a power series representation with coefficients given by the convolution of the coefficients $\{\varphi_n\}$ with themselves, i.e. the n-th coefficient in the power series $c_u(z)$ is $c_{u,n} = \sum_{j=0}^{n}\varphi_j\varphi_{n-j}$. Similarly for the cases $m_u = 3, \ldots$ one obtains the m_u-th order convolution.

iv) In the second term one finds the contribution of the positive ($\lambda_u = 0$) and negative ($\lambda_u = \pi$) real roots. The contribution of z_u is given by $B_u(L)d_u(L)\epsilon_t$, where $y_{u,t} := d_u(L)\zeta_t$, with ζ_t a univariate white noise, is the MA representation of the univariate $\text{AR}(1)^{m_u}$ process

$$(1 - w_u L)^{m_u} y_{u,t} = \zeta_t.$$

When $m_u = 1$, $d_u(L)\zeta_t$ gives either a dampened oscillation for $w_u < 0$ or a geometric decay if $w_u > 0$. We observe that the remark in *iii*) applies here substituting $c_u(z)$, φ_n with $d_u(z)$, w_n. This gives the generic coefficient of $d_u(z)$ as the m_u-th order convolution of $\{w_n\}$.

v) When $d_G \geq d_g$, one finds the additional term $R(L)\epsilon_t$ of finite degree $d_G - d_g$. Hence only the first two terms are responsible for the presence of infinite memory in X_t, i.e. for $cov(X_t, X_{t-n}) \neq 0$ for all n.

4. Relationship with the stochastic cycles of Harvey and Trimbur

In this section we discuss the relationship between univariate $\text{AR}(2)^n$ processes and the stochastic cycles of Harvey (1990), Harvey and Trimbur (2003), Trimbur (2006), see also Luati and Proietti (2009) for extensions. Both processes have the same AR polynomial, and they differ because of the presence of a MA component present in the stochastic cycles of Harvey and Trimbur (2003), which is absent in the $\text{AR}(2)^n$ processes.

The MAD representation involves $\text{AR}(2)^n$ processes $y_{n,t}$ as stochastic cycles or order n, where $y_{n,t}$ is defined by

$$\left(1 - 2\rho\cos\lambda L + \rho^2 L^2\right)^n y_{n,t} = \zeta_t \tag{4.1}$$

with ζ_t an uncorrelated univariate white noise with mean 0 and covariance matrix σ_ζ^2. The AR polynomial of (4.1) is $a(L)^n$ where $a(L)$ is the polynomial $a(z) := 1-(2\rho\cos\lambda)\,z+\rho^2 z^2 = \left(1-\rho e^{i\lambda}z\right)\left(1-\rho e^{-i\lambda}z\right)$, with two complex conjugate roots at $\rho^{-1}e^{\pm i\lambda}$.

Harvey and Trimbur (2003), building on Harvey (1990), consider the following bivariate processes $\psi_t^{[j]} := \left(\psi_{1,t}^{[j]} : \psi_{2,t}^{[j]}\right)'$

$$\psi_t^{[j]} = G\psi_{t-1}^{[j]} + S\psi_t^{[j-1]}, \qquad G := \rho\begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}, \qquad S := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.2}$$

for $j = 1, 2, \ldots, n$, where $0 \leq \lambda \leq \pi$ is a given frequency, $0 < \rho \leq 1$, and $\psi_t^{[0]} := (\kappa_{1t} : \kappa_{2t})'$ is an uncorrelated white noise with mean 0 and covariance matrix $\Sigma := \text{diag}(\sigma_1^2, \sigma_2^2)$. Because of the selection matrix S, there is no loss of generality in setting $\sigma_2^2 = 0$. They identify $\psi_{1,t}^{[n]}$ as the n-th order stochastic cycle. Trimbur (2006) studied the properties of such processes when $S = I_2$ and $\sigma_1^2 = \sigma_2^2$. We consider here the original setup (4.2) of Harvey and Trimbur (2003), with $\sigma_2^2 = 0$, which involves a single input disturbance κ_{1t}, as in the $\text{AR}(2)^n$ case.

The univariate representation of $\psi_{1,t}^{[n]}$ is an ARMA($2n$,n) of the type

$$(4.3)\qquad \left(1-2\rho\cos\lambda L+\rho^2L^2\right)^n \psi_{1,t}^{[n]}=(1-\rho\cos\lambda L)^n\,\kappa_{1,t},$$

as can be obtained by computing the final equations form of (4.2), see Harvey and Trimbur (2003)[4]. Comparing (4.1) and (4.3) one sees that $\psi_{1,t}^{[n]}$ and $y_{n,t}$ share the same AR polynomial $a(L)^n$; however, while $y_{n,t}$ has no MA polynomial, $\psi_{1,t}^{[n]}$ involves the MA(n) polynomial $(1-\rho\cos\lambda L)^n$.

Another way to discuss the relationship between $\psi_{1,t}^{[n]}$ and $y_{n,t}$ is by comparing the companion form of $y_{n,t}$ directly with the definition (4.2) of $\psi_{1,t}^{[n]}$. To this end, define $Y_t^{[j]}:=(y_{j,t}:y_{j,t-1})'$ as state vector for (4.1); one finds

$$(4.4)\qquad Y_t^{[j]}=FY_{t-1}^{[j]}+SY_t^{[j-1]},\qquad F:=\begin{pmatrix}2\rho\cos\lambda & -\rho^2\\ 1 & 0\end{pmatrix}$$

for $j=1,2,\dots,n$ with $Y_t^{[0]}:=(\zeta_t:0)'$, and covariance $\mathbb{E}(Y_t^{[0]}Y_t^{[0]\prime})=\mathrm{diag}(\sigma_\zeta^2,0)$. The following theorem shows that the matrices G in (4.2) and F in (4.4) are similar.

Theorem 4.1. *The matrices G in* (4.2) *and A in* (4.4) *are similar, i.e. $G=HFH^{-1}$ or $F=H^{-1}GH$ with*

$$H:=\left(\rho^2+1\right)^{\frac{1}{2}}\begin{pmatrix}-\left(\rho\sin\lambda\right)^{-1} & \cot\lambda\\ 0 & 1\end{pmatrix},\qquad H^{-1}=\left(\rho^2+1\right)^{-\frac{1}{2}}\begin{pmatrix}-\rho\sin\lambda & \rho\cos\lambda\\ 0 & 1\end{pmatrix},$$

where both F and G have as matrix of eigenvalues $\Lambda:=\mathrm{diag}(\rho e^{i\lambda},\rho e^{-i\lambda})$.

We observe that F, G, H are all real matrices, unlike the matrix of eigenvalues $\Lambda:=\mathrm{diag}(\rho e^{i\lambda},\rho e^{-i\lambda})$, which is complex. The following corollary shows that this implies that the generating mechanisms in (4.2) and (4.4) are related by the nonsingular transformation H above.

Corollary 4.2. *Consider the AR(2) process $y_{1,t}$ defined in* (4.1) *with companion representation* (4.4) *and the matrix H as defined in Theorem* 4.1*; then $\psi_t^{[1]}=HY_t^{[j]}$ satisfies eq.* (4.2) *with $n=1$ and*

$$(4.5)\qquad \sigma_1^2=\sigma_\zeta^2\frac{\rho^2+1}{\rho^2\sin^2\lambda},\qquad \sigma_2^2=0.$$

Vice-versa one can generate $\psi_t^{[1]}$ as in (4.2) *with* (4.5) *and set $Y_t^{[1]}=H^{-1}\psi_t^{[1]}$ which satisfies* (4.4) *for $n=1$.*

Let $H=:(H_1:H_2)'$ and $H^{-1}=:(H^1:H^2)'$, where H_j' and $H^{j\prime}$ are the j-th rows of H and H^{-1} respectively. The Corollary above implies that the bivariate generating mechanisms of $Y_t^{[1]}$ and $\psi_t^{[1]}$ correspond 1 to 1, and $\psi_{1,t}^{[1]}$ and $y_{1,t}$ can be obtained by different sampling schemes from this bivariate process. In fact, one could generate the stochastic cycle $\psi_t^{[1]}$ and obtain $\psi_{1,t}^{[1]}$ and $y_{1,t}$ as $\psi_{1,t}^{[1]}=(1:0)\psi_t^{[1]}$ and $y_{1,t}=H^{1\prime}\psi_t^{[1]}$, where $(1:0)$ and $H^{1\prime}$ are the sampling vectors. Symmetrically, one could generate the stochastic cycle $Y_t^{[1]}$ and obtain $\psi_{1,t}^{[1]}$ and $y_{1,t}$ as $\psi_{1,t}^{[1]}=H_1'Y_t^{[1]}$ and $y_{1,t}=(1:0)Y_t^{[1]}$, where H_1' and $(1:0)$ are the sampling vectors from the same bivariate process $Y_t^{[1]}$.

[4]See pag. 247 there, the lines preceding eq. (11).

The comparison of the companion form with (4.2) in the general case $n > 1$ is less straightforward. In fact, pre-multiplying (4.4) by H one finds

$$\left(HY_t^{[n]}\right) = G\left(HY_{t-1}^{[n]}\right) + HSY_t^{[n-1]} \tag{4.6}$$

where, however, $HS \neq SH$, i.e. the two matrices do not commute. If they did, then (4.6) would be equal to (4.2) when setting $\psi_t^{[n]} = HY_t^{[n]}$.

5. Spectra and impulse responses of cyclical components

In this section we describe the power spectra of the $\text{AR}(1)^n$ and $\text{AR}(2)^n$ processes

$$(1 - w_u L)^n\, y_{n,t} = \zeta_t, \tag{5.1}$$

$$\left(1 - 2\rho_u \cos\lambda_u L + \rho_u^2 L^2\right)^n y_{n,t} = \zeta_t \tag{5.2}$$

in (3.2).

Here $0 \leq \rho_u < 1$, $0 \leq |w_u| < 1$, $0 \leq \lambda_u \leq \pi$. These results are directly derived from the standard results on the spectral density of stationary univariate AR(p) case, see e.g. eq. (4.3.8) page 160 in Fuller (1996); they are reported here for completeness. We start with the $\text{AR}(1)^n$ case.

Proposition 5.1 (Spectrum of $\text{AR}(1)^n$)**.** *The spectral density $f_y(\lambda)$ of an $AR(1)^n$ process (5.1) with $|w_u| < 1$ is bounded and it is given by*

$$f_y(\lambda) = \frac{\sigma_\zeta^2}{2\pi}\left(\frac{1}{1 + w_u^2 - 2w_u \cos\lambda}\right)^n, \qquad -\pi \leq \lambda \leq \pi. \tag{5.3}$$

For $w_u = 0$ one finds the flat spectrum of a white noise process; if $w_u > 0$, the spectral density $f_y(\lambda)$ has a maximum at $\lambda = 0$ and two minima at $\lambda = \mp\pi$; if $w_u < 0$, the spectral density $f_y(\lambda)$ has two maxima at $\lambda = \mp\pi$ and a minimum at $\lambda = 0$.

The next two figures illustrate Proposition 5.1.

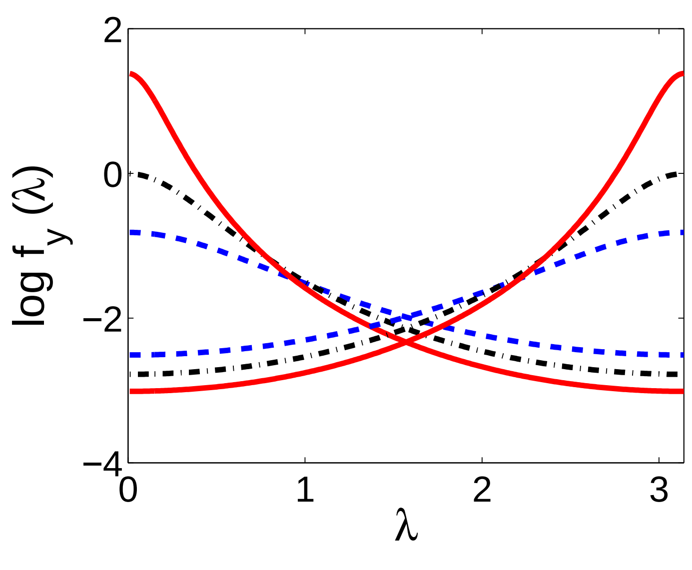

Figure 1. $\log f_y(\lambda)$ for $\text{AR}(1)^1$ processes with $w_u = \pm 0.4$ (dashed blu line), ± 0.6, (dash-dotted black lines) ± 0.8 (solid red line).

Figure 1 plots the logarithm of spectral density of six $\text{AR}(1)^1$ processes fixing $n = 1$; we note that $\log f_y(\lambda)$ has a symmetric behavior for positive and negative roots and that its peak is inversely related to the absolute value of the root, i.e. it grows with $|w_u|$. Next we fix $w_u = 0.8$ and in figure 2 we plot $\log f_y(\lambda)$ for $\mathrm{AR}(1)^n$ for three values of n; this shows that the peak is directly related to n, so that the higher n the higher the portion of spectrum on the left.

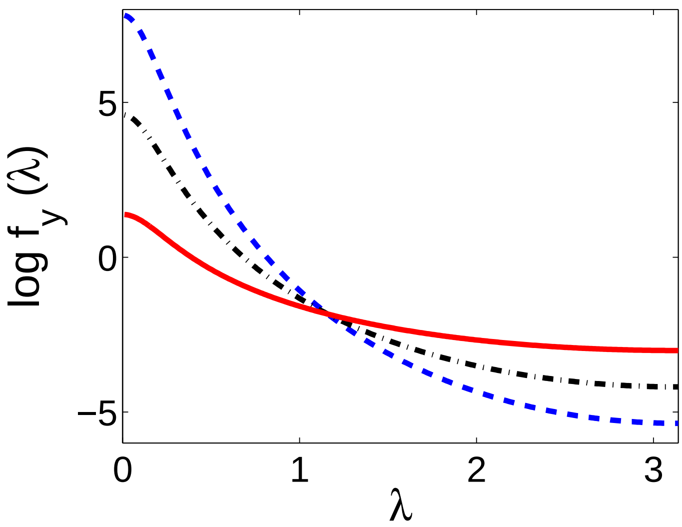

FIGURE 2. $\log f_y(\lambda)$ for $\mathrm{AR}(1)^n$ processes with $w_u = 0.8$ and $n = 3$, (dashed blu line), $n = 2$, (dash-dotted black lines) $n = 1$ (solid red line).

Similarly, we discuss the $\mathrm{AR}(2)^n$ case, see e.g. Jenkins and Watts (1969) p. 228-229 and Box and Jenkins (1976) p. 62 for the AR(2) case.

Proposition 5.2 (Spectral density of $\mathrm{AR}(2)^n$)**.** *The spectral density $f_y(\lambda)$ of an $AR(2)^n$ process* (5.2) *is bounded for $\rho_u < 1$ and it is given by*

(5.4)
$$f_y(\lambda) = \frac{\sigma_\zeta^2}{2\pi}\left(\frac{1}{1+\rho_u^2-2\rho_u\cos(\lambda-\lambda_u)}\right)^n\left(\frac{1}{1+\rho_u^2-2\rho_u\cos(\lambda+\lambda_u)}\right)^n, \qquad -\pi \le \lambda \le \pi$$

For $\rho_u = 0$ one finds the flat spectrum of a white noise process; for $0 < \rho_u < 1$, the extremum points of $f_y(\lambda)$ depend on the condition

$$\arccos\left(\frac{1}{c_u}\right) \le \lambda_u \le \arccos\left(-\frac{1}{c_u}\right), \qquad c_u := \frac{1+\rho_u^2}{2\rho_u} \ge 1. \tag{5.5}$$

If (5.5) *is satisfied, then $f_y(\lambda)$ has minima at $\lambda = 0, \mp\pi$ and maxima at $\lambda = \mp\lambda_\diamond$, where $\lambda_\diamond := \arccos(c_u\cos\lambda_u)$; in general $\lambda_\diamond \neq \lambda_u$ while $\lim_{\rho_u\to 1}\lambda_\diamond = \lambda_u$. If $\lambda_u < \arccos(c_u^{-1})$, then $f_y(\lambda)$ has a maximum at $\lambda = 0$ and minima at $\lambda = \mp\pi$; if $\arccos(-c_u^{-1}) < \lambda_u$, then $f_y(\lambda)$ has a minimum at $\lambda = 0$ and maxima at $\lambda = \mp\pi$.*

A few remarks are in order.

i) The extrema of the spectral density $f_y(\lambda)$ in (5.4) do not depend on n. More precisely, n influences only the scale and location of $\log f_y(\lambda)$, but not its form. In fact $\log f_y(\lambda) = a + n\ell(\lambda)$, where $a := \log\sigma_\zeta^2 - \log 2\pi$ does not depend on λ, n and $\ell(\lambda) := \log h(\lambda-\lambda_u) + \log h(\lambda+\lambda_u)$, $h(\varphi,\rho) := 1 - 2\rho\cos\varphi + \rho^2$.

ii) The location of the maximum of the spectral density $f_y(\lambda)$ depends both on λ_u and ρ_u; the dependence on ρ_u involves the condition (5.5) and the coefficient c_u which regulates the diplacement from λ_u in $\lambda_\diamondsuit := \arccos(c_u \cos\lambda_u)$, which is defined if this condition holds.

iii) When $\lambda_u = \pi/2$, the spectral density becomes

$$f_y(\lambda) = \frac{\sigma_\zeta^2}{2\pi}\left(\frac{1}{(1+\rho_u^2)^2 - 4\rho_u^2 \sin^2(\lambda)}\right)^n$$

which is symmetric also over the segment $0 \leq \lambda \leq \pi$ (respectively $-\pi \leq \lambda \leq 0$) about the line $\lambda = \pi/2$ ($\lambda = -\pi/2$) with maxima at $\mp\pi/2$.

iv) c_u is a monotonically decreasing function of ρ_u. As ρ_u becomes small, one finds large values of c_u and $\arccos(c_u^{-1})$ and $\arccos(-c_u^{-1})$ both get close to $\pi/2$; for fixed λ_u this implies that the condition (5.5) eventually fails as ρ_u becomes small, and one has extrema only at $\lambda = 0$ and $\lambda = \mp\pi$, where the location of the min and max depends on where λ_u is with respect to $\pi/2$. When ρ_u increases to 1, c_u tends to 1 as well and the condition (5.5) approaches $-\pi \leq \lambda_u \leq \pi$, which will be eventually met for any $\lambda_u \neq 0, \mp\pi$. At the same time the diplacement due to c_u disappears and $\lim_{\rho_u \to 1} \lambda_\diamondsuit = \lambda_u$.

v) The displacement $c_u \geq 1$, so that $\lambda_\diamondsuit$ is closer to 0 than λ_u.

The next two figures illustrate Proposition 5.2.

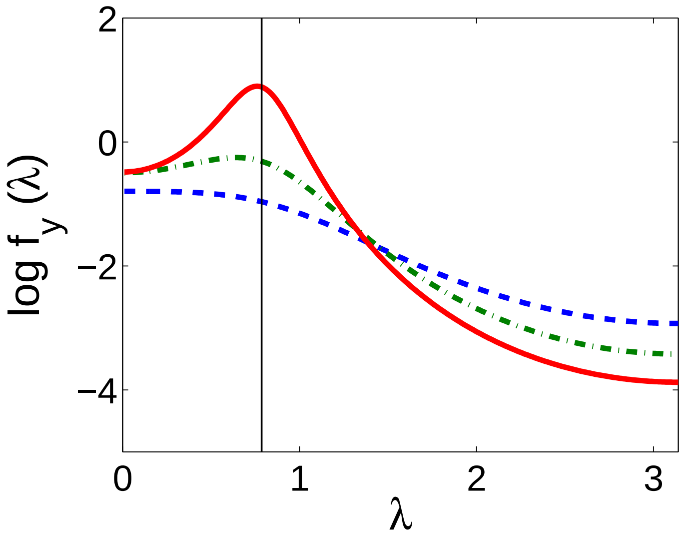

FIGURE 3. $\log f_y(\lambda)$ for $AR(2)^1$ processes with $\lambda_u = \pi/4$ and $\rho_u = 0.4$ (dashed blu line), 0.6, (dash-dotted green line) 0.8 (solid red line). The vertical line is at $\lambda = \lambda_u = \pi/4$.

Figure 3 plots $\log f_y(\lambda)$ for three $AR(2)^1$ processes fixing $n = 1$ and $\lambda_u = \pi/4$; we note that the peak is not at λ_u (vertical line) and that it grows with $|w_u|$. Next we fix $\rho_u = 0.6$ and in figure 4 we plot $\log f_y(\lambda)$ for $AR(2)^n$ for three values of n; this shows that n does not affect the displacement of the peak and that the higher n the higher the portion of spectrum on the left.

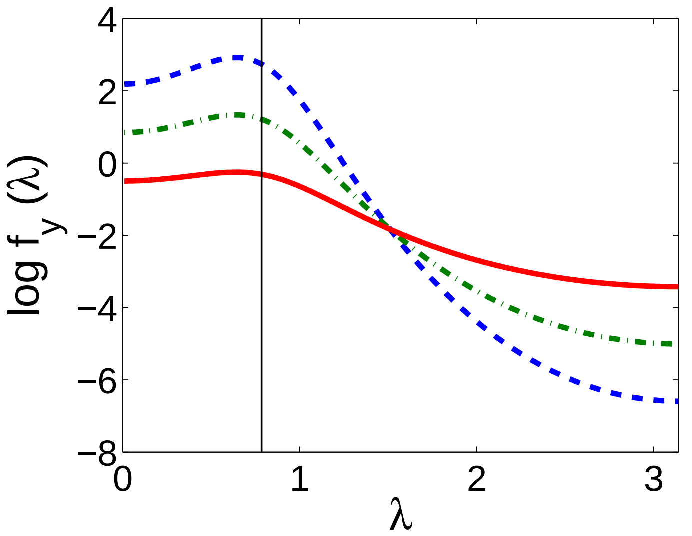

FIGURE 4. $\log f_y(\lambda)$ for $\mathrm{AR}(2)^n$ processes with $\lambda_u = \pi/4$ and $\rho_u = 0.6$ and $n = 3$ (dashed blu line), $n = 2$, (dash-dotted green line) $n = 1$ (solid red line). The vertical line is at $\lambda = \lambda_u = \pi/4$.

Next we illustrate the univariate impulse response functions (IRF) $c_u(z)$, $d_u(z)$ in (3.3).

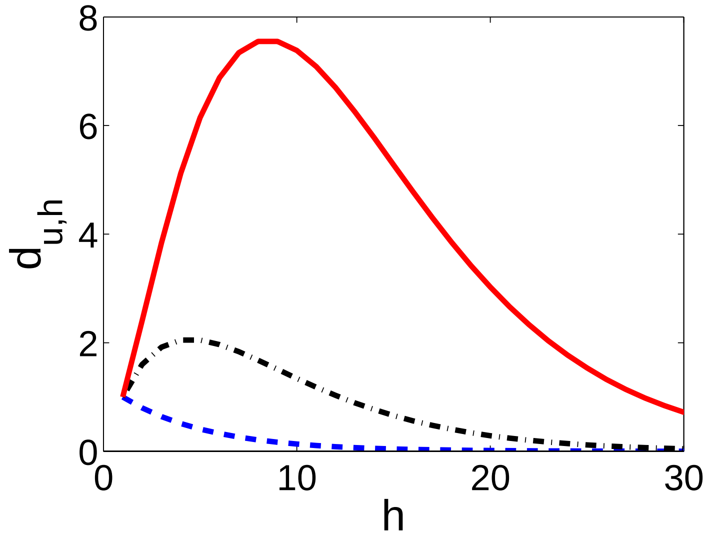

FIGURE 5. IRF $d_{u,h}$ for $\mathrm{AR}(1)^n$ processes with $w_u = 0.8$ and $n = 1$ (dashed blu line), $n = 2$, (dash-dotted black lines) $n = 3$ (solid red line).

Figure 5 plots $d_{u,h}$ in $d_u(z) =: \sum_{h=0}^{\infty} d_{u,h} z^h$ for three $\mathrm{AR}(2)^n$ processes fixing $w_u = 0.8$; the coefficients of the expansion decrease exponentially only for $n = 1$ while they display a peak that moves on the right and increases magnitude as n grows. A similar behavior applies for $w_u < 0$ and for complex roots, see figure 6 and 7 in the next page.

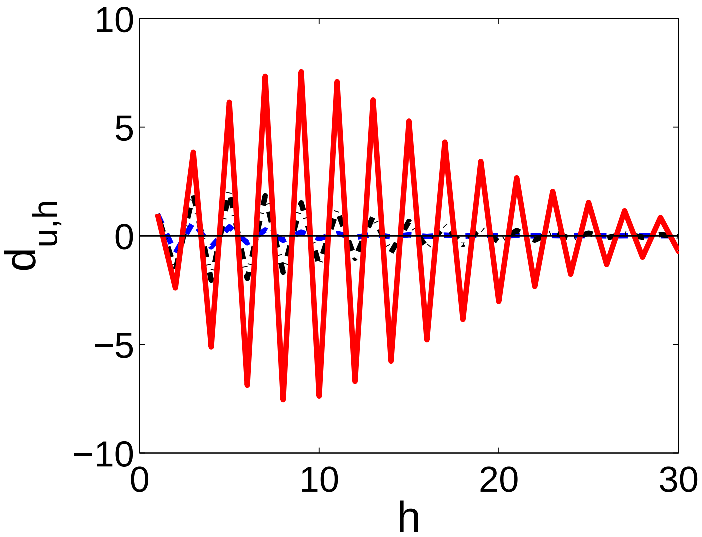

FIGURE 6. IRF $d_{u,h}$ for AR$(1)^n$ processes with $w_u = -0.8$ and $n = 1$ (dashed blu line), $n = 2$, (dash-dotted black lines) $n = 3$ (solid red line).

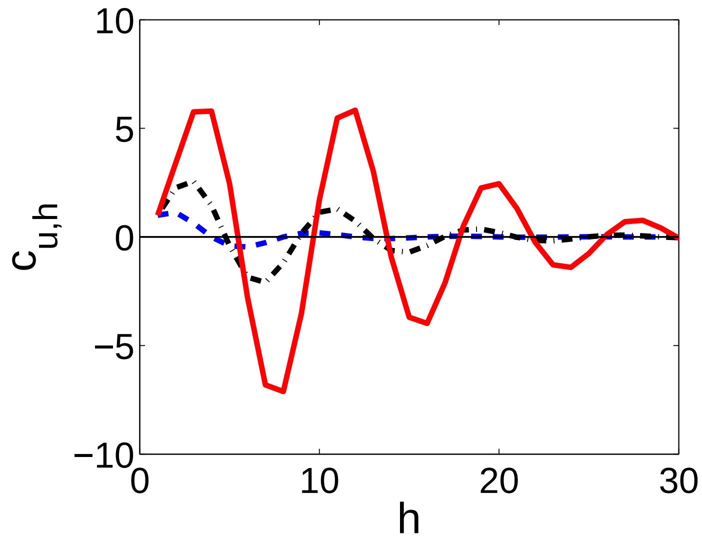

FIGURE 7. IRF $c_{u,h}$ for AR$(2)^n$ processes with $\lambda_u = \pi/4$, $\rho_u = 0.8$ and $n = 1$ (dashed blu line), $n = 2$, (dash-dotted black lines) $n = 3$ (solid red line).

6. Properties of the loadings in the MAD representation

In this section we describe the left-null space structure of the loadings $A_u(z)$, $B_u(z)$ in (3.2); this is useful to characterize linear combinations of X_t which have specific spectral properties.

First we introduce a procedure called 'polynomial rank factorization' of a matrix polynomial at a given point. Let $A(z)$ be a square matrix polynomial whose inverse has a pole of order m at $z = w$; the polynomial rank factorization Definition 6.1 below gives a characterization of the set of reduced rank restrictions that are satisfied by the coefficients of a matrix polynomial whose inverse function has a pole of given order at a specific point. That is, if $A(z)$ and its derivatives at $z = w$ satisfy those conditions then inv $A(z)$ has a pole of order m at the same point; the converse is also true, i.e. if inv $A(z)$ has a pole of order m at $z = w$ then $A(z)$ and its derivatives at $z = w$ satisfy the rank restrictions of the polynomial rank factorization at that point. Hence the polynomial rank factorization is a one to one and onto map from the structure of the matrix polynomial to the order of the pole of its inverse function which reveals the relevant subspaces that characterize its reduced rank structure. This result is based on the recursive algorithm developed in Franchi (2009) and further analyzed in Franchi and Paruolo (2009).

Definition 6.1 (Polynomial rank factorization of $A(z)$ at $z = w$)**.** *Let $A(z) = \sum_{n=0}^{d_A} A_n(z-w)^n$ be a square matrix polynomial whose inverse has a pole of order m at $z = w$; define α_0 and β_0 from the matrix rank factorization*

$$A_0 = -\alpha_0\beta_0' \tag{6.1}$$

and for $j = 1, \dots, m$, let $a_j := (\alpha_0 : \cdots : \alpha_{j-1})$, $b_j := (\beta_0 : \cdots : \beta_{j-1})$ define α_j and β_j from the matrix rank factorization

$$M_{a_j} A_{j,1} M_{b_j} = -\alpha_j\beta_j', \tag{6.2}$$

where $A_{j,k}$ is defined for $j, k \geq 1$ from the recursions

$$A_{j,k} := A_{j-1,k+1} + A_{j-1,1}\sum_{n=0}^{j-2}\bar{\beta}_n\bar{\alpha}_n' A_{n+1,k} \tag{6.3}$$

with initial values $A_{0,k} := A_{k-1}$.

The following additional remarks are in order:

Remark 6.2. *Eq.* (6.1), (6.2) *define α_j, β_j up to a conformable change of bases of the row and column spaces; this does not affect the results.*

Remark 6.3. *The square matrices $(\alpha_0 : \cdots : \alpha_{m_u})$ and $(\beta_0 : \cdots : \beta_{m_u})$ are non-singular with orthogonal blocks, i.e. $\alpha_j'\alpha_k = \beta_j'\beta_k = 0$ for $j \neq k$.*

Remark 6.4. *The conditions* (6.2) *are reduced-rank conditions for $j = 1, \cdots, m-1$, while the terminal condition for $j = m$ is a full-rank condition.*

Remark 6.5. *For $w = 1$, and $m = 1$, $m = 2$ these conditions were derived by Johansen (1992) and are called the $I(1)$ and $I(2)$ conditions.*

Remark 6.6. *There is a duality between the polynomial rank factorizations of $A(z)$ and of its (reduced) adjoint $H(z)$, say; let α_j, β_j and ξ_j, η_j be respectively defined by the polynomial rank factorization of $A(z)$ and $H(z)$ at $z = w$; then for $j = 0, \dots, m$, one has*

$$\operatorname{col}\xi_j = \operatorname{col}\beta_{m-j} \qquad \textit{and} \qquad \operatorname{col}\eta_j = \operatorname{col}\alpha_{m-j}, \tag{6.4}$$

see Franchi and Paruolo (2009) for the proof.

Remark 6.7. *In the following we let $a_{u,j} := (\alpha_{u,0} : \cdots : \alpha_{u,j-1})$, $b_{u,j} := (\beta_{u,0} : \cdots : \beta_{u,j-1})$ and $\Pi_{j,k}^{(u)}$ be defined by the polynomial rank factorization of $\Pi(z)$ at $z = z_u$, so that the point at which the procedure is conducted is explicitely referenced.*

Next we present relations among the coefficients of the polynomial rank factorization of $\Pi(z)$ at $z = z_u$ and the left-null space structure of the loadings, starting from the leading case with $m_u = 1$, $u = 1, \dots, q$.

Theorem 6.8 (Left-null space structure of the loadings when $m_u = 1$)**.** *Let $m_u = 1$, $u = 1, \ldots, q$; then*

$$X_t = \sum_{u\,:\,0<\lambda_u<\pi} (A_{u,0} + A_{u,1}L)c_u(L)\epsilon_t + \sum_{u\,:\,\lambda_u\in\{0,\pi\}} h_u B_{u,0} d_u(L)\epsilon_t + hR(L)\epsilon_t,$$

where

$$\gamma' B_{u,0} = 0 \Leftrightarrow \operatorname{col}\gamma \subseteq \operatorname{col}\beta_{u,0}$$

and

$$\gamma'(A_{u,0} : A_{u,1}) = 0 \Leftrightarrow \operatorname{col}(\gamma) \subseteq \operatorname{col}(\operatorname{Re}\beta_{u,0} : \operatorname{Im}\beta_{u,0}),$$

where $\alpha_{u,0}\beta'_{u,0} := \Pi(z_u)$.

Because $\operatorname{col} B_{u,0} = \operatorname{col}\beta_{u,1}$, the contribution of the root z_u is always absent from the linear combination $\beta'_{u,0}X_t$; when the root is complex this however leads to a complex process. When $\mathcal{A} := \operatorname{col}\operatorname{Re}\beta_{u,0} \cap \operatorname{col}\operatorname{Im}\beta_{u,0} \neq \{0\}$ it is possible to find a real linear combination $\gamma' X_t$ that eliminates the contribution of the complex pair z_u, z_u^* by letting $\gamma \subset \mathcal{A}$.

In general, using the coefficients defined by the polynomial rank factorization, we construct a matrix polynomial $\gamma_u(z)$ such that $\gamma'_u(L)X_t$ does not contain the contribution of the root z_u, see Theorem 6.9 below.

Theorem 6.9 (Left-null space structure of the loadings)**.** *Let $\alpha_{u,j}$, $\beta_{u,j}$ be defined by the polynomial rank factorization of $\Pi(z) = \sum_{n=0}^{d_\Pi} \Pi_{u,n}(z - z_u)^n$ at $z = z_u$ and*

$$\gamma'_u(z) := \beta'_{u,0} - \bar{\alpha}'_{u,0} \sum_{n=1}^{m_u-1} (-z_u)^n \Pi_{u,n}(1 - w_u z)^n; \tag{6.5}$$

then

$$\gamma'_u(L)X_t$$

does not contain the contribution of z_u.

Note that when the root is complex this however leads to a complex process. The conditions under which it is possible to eliminate the contribution of the complex pair z_u, z_u^* are still work in progress.

7. Examples: polynomial rank factorization and MAD representation

7.1. Example 1.

Here we illustrate how to calculate the MAD representation of a VAR and illustrates the duality result in (6.4); consider

$$X_t = \begin{pmatrix} -1 & -4/3 \\ 2 & 5/3 \end{pmatrix} X_{t-1} + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X_{t-2} + \epsilon_t;$$

then

$$\det \Pi(z) = -\frac{1}{3}(2z - 3)$$

and

$$\operatorname{adj}\Pi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -5/3 & -4/3 \\ 2 & 1 \end{pmatrix} z - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} z^2.$$

Hence there is only one characteristic root, it has multiplicity one, i.e. $w_1 = \frac{2}{3}$ and $m_1 = 1$, and the difference between the degrees of $\operatorname{adj}\Pi(z)$ and $\det\Pi(z)$ equals 1, which implies $d_R = 1$.

The polynomial rank factorization of $\Pi(z)$ at $z = \frac{3}{2}$ gives

$$\alpha_0 = \frac{1}{8}(1 : -3)', \quad \beta_0 = -(11 : 7)', \quad \alpha_1 = \frac{4}{1275}(3 : 1)', \quad \beta_1 = (-7 : 11)'$$

and that of $G(z)$ at the same point

$$\xi_0 = \frac{1}{8}(-7 : 11)', \quad \eta_0 = -(3 : 1)', \quad \xi_1 = \frac{4}{1275}(11 : 7)', \quad \eta_1 = (1 : -3)';$$

this illustrates the duality result in (6.4). Then the MAD representation is

$$X_t = B \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \epsilon_{t-n} + R_0 \epsilon_t + R_1 \epsilon_{t-1}, \tag{7.1}$$

where $B = \frac{1}{8}\begin{pmatrix} -7 \\ 11 \end{pmatrix}(3 : 1)$, $R_0 = \frac{1}{8}\begin{pmatrix} 29 & 7 \\ -33 & -3 \end{pmatrix}$ and $R_1 = \frac{3}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Note that (7.1) can be written as

$$X_t = \begin{pmatrix} -7 \\ 11 \end{pmatrix} c_t + R_0 \epsilon_t + R_1 \epsilon_{t-1},$$

where $c_t := \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n u_{t-n}$ is univariate because $u_t := \frac{1}{8}(3 : 1)\epsilon_t$.

The next two figures illustrate (7.1).

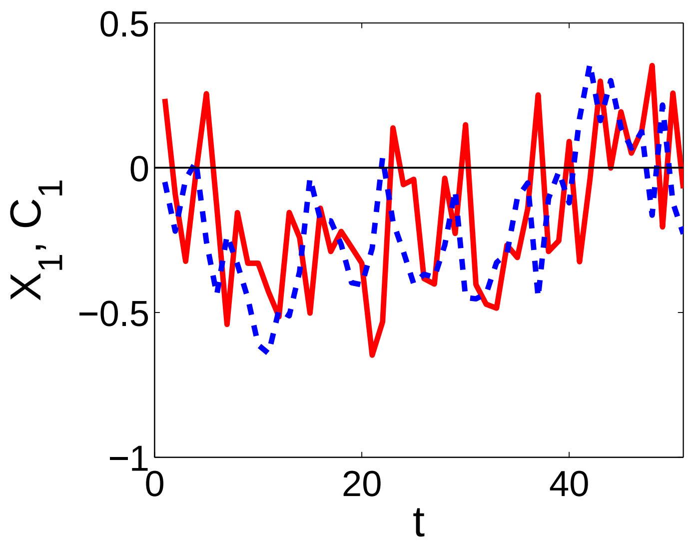

FIGURE 8. 50 realizations of the first variable in the VAR(2) process in (7.1). Red line: $X_{1,t}$ time series, blue dashed line: $C_{1,t}$ time series of the cycle. Both series are scaled by their sample range.

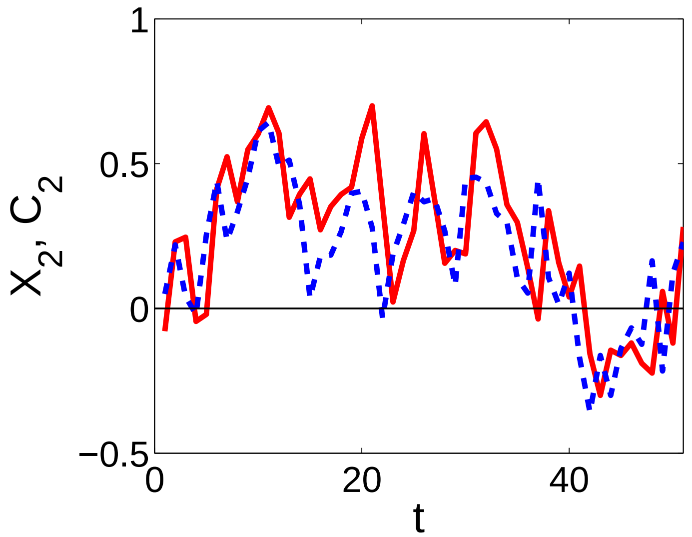

FIGURE 9. 50 realizations of the second variable in the VAR(2) process in process in (7.1). Red solid line: $X_{2,t}$ time series, blue dashed line: $C_{2,t}$ time series of the cyclical component (the same series used in Fig. 8. up to sign). Both series are scaled by their sample range.

7.2. **Example 2.** Here we discuss the iterpretation of the MAD representation using the VAR in Benati and Surico (2009),

$$X_t = \begin{pmatrix} 1.21 & 0.01 & 0.14 \\ -0.03 & 0.47 & 0.07 \\ -0.11 & -0.05 & 1.02 \end{pmatrix} X_{t-1} + \begin{pmatrix} -0.32 & -0.01 & -0.05 \\ 0.02 & -0.02 & -0.02 \\ 0.08 & 0.00 & -0.23 \end{pmatrix} X_{t-2} + \epsilon_t;$$

because

$$\det \Pi(z) = c_z(z-1.24)(z-1.57)(z-2.18)(z-2.38)(z-2.95)(z-20.95),$$

each the characteristic root z_u is real and it has multiplicity $m_u = 1$. Moreover, because $\deg \operatorname{adj} \Pi(z) = 4$ and $\deg \det \Pi(z) = 6$, their difference is negative and the finite MA part $R(L)\epsilon_t$ is absent from the MAD, see Remark v) below Theorem 3.3.

By computing the polynomial rank factorization of $\Pi(z)$ at z_u, we define $\alpha_{u,j}$, $\beta_{u,j}$ for $u = 1, \ldots, 6$ and $j = 0, 1$; the MAD representation is

$$X_t = \sum_{u=1}^{q} B_u d_u(L)\epsilon_t, \tag{7.2}$$

where $B_u = -w_u \bar{\beta}_{u,1} \bar{\alpha}'_{u,1}$ and $d_u(z) = \sum_{n=0}^{\infty} w_u^n z^n$, $w_u := 1/z_u$. Here one has

$$B_1 = \begin{pmatrix} 1 \\ -0.03 \\ -0.13 \end{pmatrix} (1.78 : -0.27 : 1.87), B_2 = \begin{pmatrix} 1 \\ -0.22 \\ -1.18 \end{pmatrix} (0.61 : 0.69 : -2.76),$$

$$B_3 = \begin{pmatrix} 1 \\ -1.44 \\ -2.28 \end{pmatrix} (-0.51 : -0.71 : 0.58), B_4 = \begin{pmatrix} 1 \\ 0.7 \\ -0.87 \end{pmatrix} (-0.97 : 0.25 : 0.28),$$

$$B_5 = \begin{pmatrix} 1 \\ 1.8 \\ -14.4 \end{pmatrix} (0.07 : 0.04 : 0.03), B_6 = \begin{pmatrix} 1 \\ -31.5 \\ 0.82 \end{pmatrix} \frac{1}{1000}(0.16 : 4 : -0.4).$$

Hence (7.2) can be written as

$$X_t = \begin{pmatrix} 1 \\ -0.03 \\ -0.13 \end{pmatrix} c_{1,t} + \begin{pmatrix} 1 \\ -0.22 \\ -1.18 \end{pmatrix} c_{2,t} + \begin{pmatrix} 1 \\ -1.44 \\ -2.28 \end{pmatrix} c_{3,t} +$$

$$\begin{pmatrix} 1 \\ 0.7 \\ -0.87 \end{pmatrix} c_{4,t} + \begin{pmatrix} 1 \\ 1.8 \\ -14.4 \end{pmatrix} c_{5,t} + \begin{pmatrix} 1 \\ -31.5 \\ 0.82 \end{pmatrix} c_{6,t},$$

where $c_{u,t} := \sum_{n=0}^{\infty} w_u^n u_{t-n}$ is univariate because $u_t := -w_u \bar{\alpha}'_{u,1} \epsilon_t$. Note that the MAD representation is invariant with respect to the orthogonalization of the shocks; in fact, for $\epsilon_t = D\xi_t$ and $Var(\xi_t) = I$, one has $u_t := -w_u \bar{\alpha}'_{u,1} D\xi_t$ and $Var(u_t) = w_u^2 \bar{\alpha}'_{u,1} DD' \bar{\alpha}_{u,1} = w_u^2 \bar{\alpha}'_{u,1} Var(\epsilon_t) \bar{\alpha}_{u,1}$.

The next figure plots $c_{1,t}, \ldots, c_{6,t}$ in (7.2).

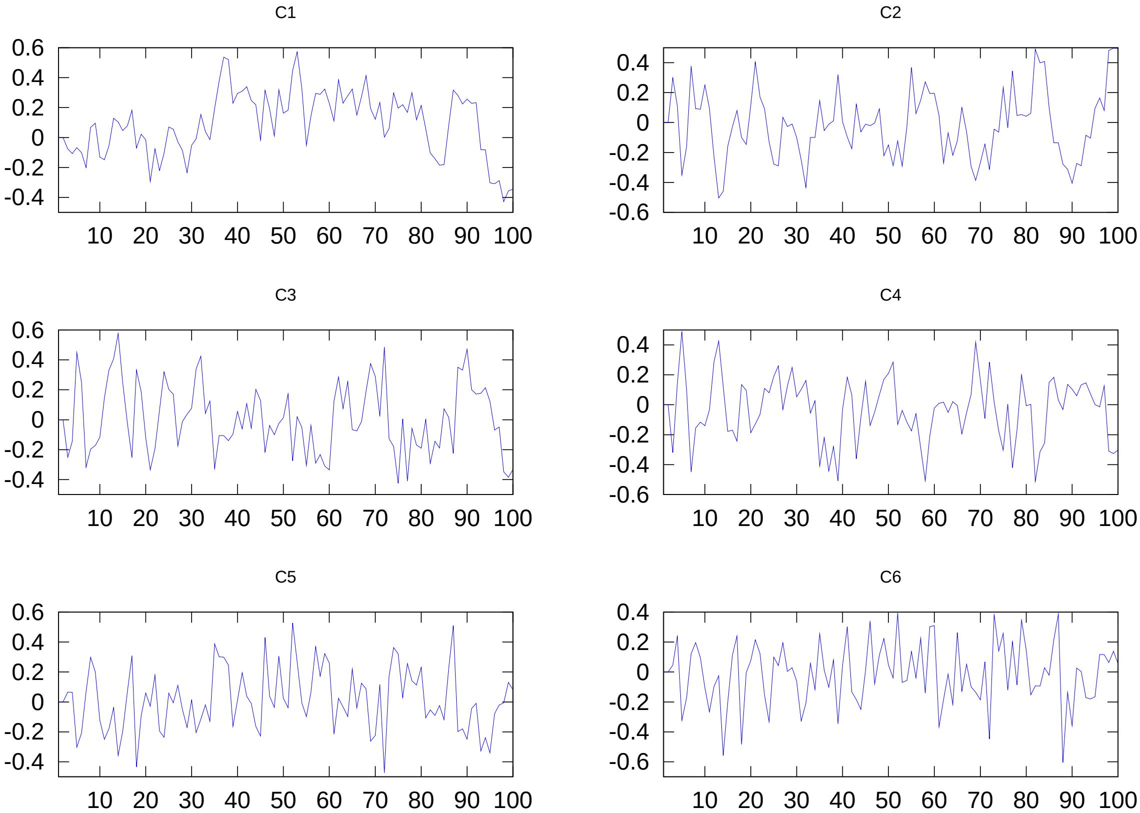

FIGURE 10. 100 realizations of the six stochastic cycles.

Next we associate cycles to variables.

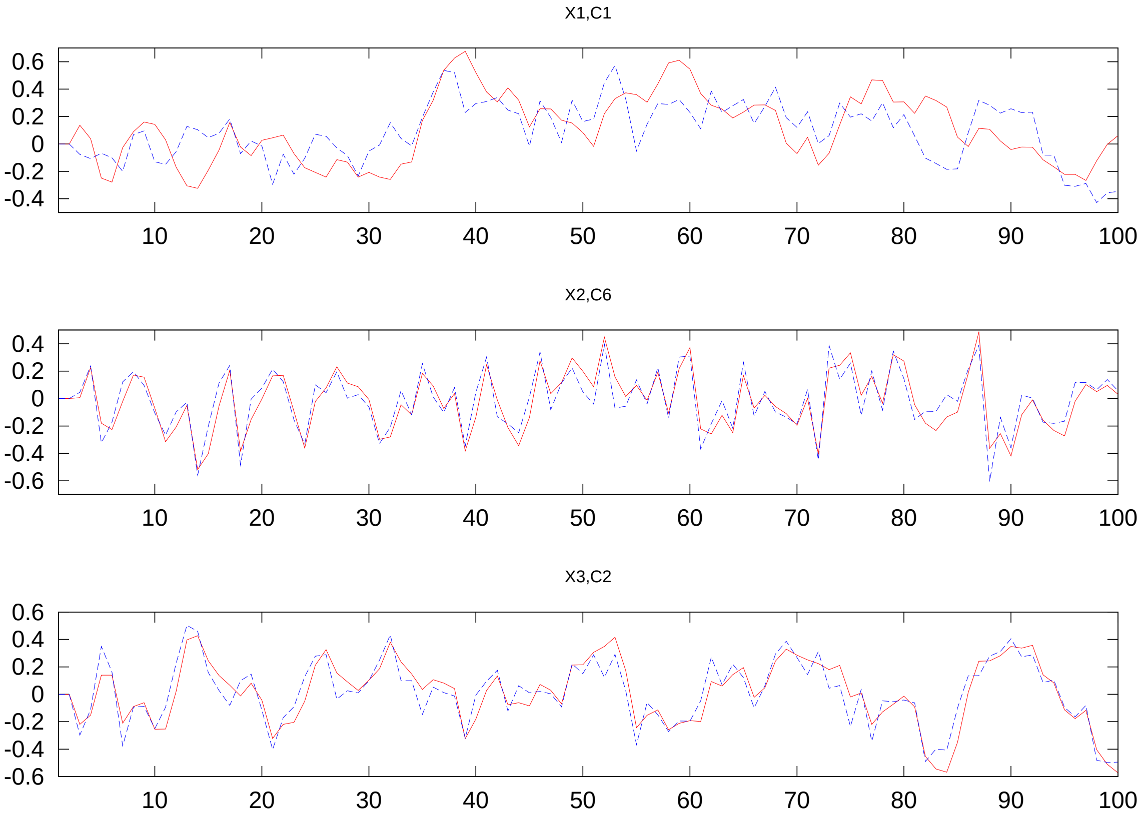

Figure 11. Association of cycles and variables. Red line: $X_{j,t}$ time series, blue line: $C_{u,t}$ time series of the cycle. Both series are scaled by their sample range.

8. Conclusions

The MAD representation provides a decomposition of a VAR process into cyclical components, which are closely connected with existing definitions of stochastic cycles. Cancelations of the cyclical components by linear combinations and/or by filtering of the observed series is discussed.

Appendix A. Proofs

Proof of Lemma 3.1. Because $\det\Pi(z_u) = 0$ one has $0 \le \operatorname{rank}\Pi(z_u) \le p-1$; when $0 \le \operatorname{rank}\Pi(z_u) < p-1$ one has $\operatorname{adj}\Pi(z_u) = 0$ and thus each entry of $\operatorname{adj}\Pi(z)$ contains the factor $(1-z/z_u)^{b_u}$ for some $0 < b_u < a_u$; if $\operatorname{rank}\Pi(z_u) = p-1$ then $\operatorname{adj}\Pi(z_u) \ne 0$ and thus $b_u = 0$. If $\operatorname{Im} z_u \ne 0$ then the same applies to z_u^*. Let $g(z) := \prod_{u=1}^q (1-z/z_u)^{m_u} =: (1-z/z_u)^{m_u} g_u(z)$; because $\operatorname{inv}\Pi(z) := \frac{\operatorname{adj}\Pi(z)}{\det\Pi(z)}$ and $G(z_u), g_u(z_u) \ne 0$ one has the last statement. □

Proof of Lemma 3.2. We observe that $p_u(z_j) = 0$ for $u \ne j$ and $B_u(z_u) = B_{u,0} = P(z_u)/p_u(z_u)$; this implies that $v(z) := P(z) - B(z) = P(z) - \sum_{u=1}^q p_u(z)B_u(z)$ is zero for $z = z_u$, for all $u = 1, \dots, q$. Hence $v(z)$ contains $a(z) := \prod_{u=1}^q (z - z_u)$ as a factor, i.e. $v(z) = a(z)R_a(z)$.

Note next that $p_u(z)$ contains the factor $(z-z_j)^{\ell_j}$ for $u \ne j$ so that $p_u^{(n)}(z_j) = 0$ for $n = 0, \dots, \ell_j - 1$. This factor appears in the derivative of $B(z)$ of order n evaluated at $z = z_j$,

$$B^{(n)}(z_j) = \sum_{u=1}^q (p_u B_u)^{(n)}(z_j) = \sum_{u=1}^q \sum_{w=0}^n \binom{n}{w} p_u^{(w)}(z_j) B_u^{(n-w)}(z_j) = \sum_{w=0}^n \binom{n}{w} p_j^{(w)}(z_j) B_j^{(n-w)}(z_j)$$

where we have used Leibniz rule on the derivative of a product in the second equality. By definition $B_j^{(n-w)}(z_j) = (n-w)! B_{j,n-w} = H_j^{(n-w)}(z_j)$, so that, applying Leibniz rule backwards to $\sum_{w=0}^n \binom{n}{w} p_j^{(w)}(z_j) B_j^{(n-w)}(z_j) = \sum_{w=0}^n \binom{n}{w} p_j^{(w)}(z_j) H_j^{(n-w)}(z_j) = (p_j H_j)^{(n)}(z_j)$, one finds $B^{(n)}(z_j) = (p_j H_j)^{(n)}(z_j) = P^{(n)}(z_j)$. This proves $B^{(n)}(z_j) = P^{(n)}(z_j)$ and $v^{(n)}(z_j) = 0$ for $n = 0, \dots, \ell_j - 1$ and $j = 1, \dots, q$, so that $v(z) = p(z)R(z)$.

Next observe that the degree of $p_u(z)$ is $d_{p_u} = d_p - \ell_u$, where $d_p := \sum_{u=1}^q \ell_u$ and the degree of $B_u(z)$ is $d_{B_u} = \ell_u - 1$ so that the degree of $p_u(z)B_u(z)$ is $d_p - 1$; this implies that also the degree of $B(z)$ is $d_B = d_p - 1$. Because $\deg v(z) = \deg p(z)R(z) = d_P$, one has $d_R = d_P - d_p$. This completes the proof. □

The next lemma is used in the proof of Theorem 3.3 below.

Lemma A.1. *Let $z_v = z_u^*$; then $B_v(z) = B_u(z)^*$.*

Proof. By definition $B_u(z) = \sum_{s=0}^{m_u-1} B_{u,s}(z - z_u)^s$ where $B_{u,s} := (s!)^{-1} G_u^{(s)}(z_u)$, $G_u(z) := H(z)/h_u(z)$. Define $1/h_u(z) =: s_u^{-1}(z)\, q_v(z)$, $q_v(z) := (z - z_v)^{-m_v}$, $s_u(z) = \prod_{j \ne u,v} (z - z_j)^{m_j}$, $K_u(z) := H(z) s_u^{-1}(z)$, so that $G_u(z) = K_u(z)\, q_v(z)$. Apply Leibniz' rule to find

$$B_{u,s} := (s!)^{-1} G_u^{(s)}(z_u) = (s!)^{-1} \sum_{j=0}^s \binom{s}{j} K_u^{(j)}(z_u)\, q_v^{(s-j)}(z_u).$$

Similarly one finds

$$B_{v,s} := (s!)^{-1} G_v^{(s)}(z_v) = (s!)^{-1} \sum_{j=0}^s \binom{s}{j} K_v^{(j)}(z_v)\, q_u^{(s-j)}(z_v).$$

We note that $q_v^{(s-j)}(z_u)$ is a (real) function of $z_u - z_v$, which is a real number, so that $q_v^{(s-j)}(z_u) = q_u^{(s-j)}(z_v)$, and that $s_u(z) = s_v(z)$, so that $K_u(z) = K_v(z)$; this implies $K_v^{(j)}(z_v) = K_u^{(j)}(z_u)^*$ and hence $B_{u,s} = B_{v,s}^*$. □

Proof of Theorem 3.3. Let $z_v = z_u^*$ and define $A_u(z) := (1 - w_v z)^{m_u} h_u B_u(z) + (1 - w_u z)^{m_u} h_v B_v(z)$; hence by Lemma A.1 $A_u(z)$ has real coefficients and grouping the complex pairs together, (3.1) becomes

$$\operatorname{inv}\Pi(z) = \sum_{u\,:\,0<\lambda_u<\pi} \frac{A_u(z)}{(1-w_u z)^{m_u}(1-w_u^* z)^{m_u}} + \sum_{u\,:\,\lambda_u\in\{0,\pi\}} \frac{c_u B_u(z)}{(1-w_u z)^{m_u}} + hR(z).$$

For $0 < \lambda_u < \pi$ one has

$$c_u(z) := \frac{1}{(1-w_u z)^{m_u}(1-w_u^* z)^{m_u}} = \left(\sum_{n=0}^{\infty} \frac{\sin(n+1)\lambda_u}{\sin\lambda_u}\rho_u^n z^n\right)^{m_u}$$

and for $\lambda_u \in \{0,\pi\}$, $w_u = \rho_u$ so that

$$d_u(z) := \frac{1}{(1-w_u z)^{m_u}} = \left(\sum_{n=0}^{\infty} \rho_u^n z^n\right)^{m_u}.$$

□

Proof of Theorem 4.1. The eigenvalue decomposition of the companion matrix F is $FV = V\Lambda$ with $\Lambda = \operatorname{diag}(\rho e^{i\lambda}, \rho e^{-i\lambda})$ and

$$V := \left(\rho^2+1\right)^{-\frac{1}{2}} \begin{pmatrix} \rho e^{i\lambda} & \rho e^{-i\lambda} \\ 1 & 1 \end{pmatrix}, \qquad V^{-1} = \frac{(\rho^2+1)^{\frac{1}{2}}}{2i\sin\lambda} \begin{pmatrix} \rho^{-1} & -e^{-i\lambda} \\ -\rho^{-1} & e^{i\lambda} \end{pmatrix},$$

and the one of the matrix G is $GU = U\Lambda$ with

$$U = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \qquad U^{-1} := \frac{1}{2}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}.$$

Hence $V^{-1}FV = \Lambda = U^{-1}GU$, from which $G = HFH^{-1}$ or $F = H^{-1}GH$ for $H := UV^{-1}$ with

$$H := UV^{-1} = \frac{(\rho^2+1)^{\frac{1}{2}}}{2i\sin\lambda} \begin{pmatrix} -2i\rho^{-1} & 2i\cos\lambda \\ 0 & 2i\sin\lambda \end{pmatrix} = \left(\rho^2+1\right)^{\frac{1}{2}} \begin{pmatrix} -\left(\rho\sin\lambda\right)^{-1} & \cot\lambda \\ 0 & 1 \end{pmatrix}.$$

□

Proof of Corollary 4.2. From Theorem 4.1 we see that $HY_t^{[1]} = (HFH^{-1})HY_{t-1}^{[1]} + HSY_t^{[0]}$, where $HSY_t^{[0]} = HY_t^{[0]}$ has covariance

$$\mathbb{E}(HY_t^{[0]}Y_t^{[0]\prime}H') = \sigma_1^2 H \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} H' = \sigma_1^2 \frac{\rho^2+1}{\rho^2\sin^2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

□

Proof of Proposition 5.1. Apply eq. (4.3.8) page 160 in Fuller (1996) with coincident roots, i.e. m_j, ω, p equal to w_u, λ, n in the present notation; this gives (5.3). Define the function $g(\lambda) := h(\lambda, w_u) := a_u - b_u\cos\lambda$, $a_u := 1 + w_u^2$, $b_u := 2w_u$ and note that $f_y(\lambda)$ has a min (max) when $g(\lambda)$ has a max (min). Moreover $f_y(\lambda)$ is bounded when $h(\lambda, w_u) > 0$, which is always the case for $|w_u| < 1$ by Lemma A.2.

Let $\dot{g}$, $\ddot{g}$ indicate the first and second derivative of g wrt λ; one finds $\dot{g}(\lambda) = b_u\sin\lambda$, $\ddot{g}(\lambda) = b_u\cos\lambda$, so that $\dot{g}(\lambda) = 0$ for $\lambda = 0, \mp\pi$ and $\ddot{g}(0) = b_u$, $\ddot{g}(\pm\pi) = -b_u$. This shows that if $w_u > 0$, then $b_u > 0$ and $g(\lambda)$ (respectively $f_y(\lambda)$) has a min (max) at $\lambda = 0$ and two maxima (minima) at $\lambda = \mp\pi$; if $w_u < 0$, then $b_u < 0$ and $g(\lambda)$ ($f_y(\lambda)$) has two maxima (minima) at $\lambda = \mp\pi$ and a min (max) at $\lambda = 0$. □

For the proof of of Proposition 5.2 below we employ the following result.

Lemma A.2. *The function* $h(\varphi,\rho) := a - b\cos\varphi = b(c-\cos\varphi)$*, where* $a = 1+\rho^2$*,* $b = 2\rho$*,* $0 < |\rho| < 1$ *is non-negative and* $h(\varphi,\rho)$ *has roots at* $(\varphi,\rho) = (0,1)$ *and* $(\mp\pi,-1)$*.*

Proof. Because $\rho, b \neq 0$, define $c := a/b$ and note that $|c| \geq 1$, because

$$|c| - 1 = \frac{a-|b|}{|b|} = \frac{1+\rho^2-2|\rho|}{|b|} = \frac{(1-|\rho|)^2}{|b|} \geq 0,$$

where the inequality is strict except for $\rho = 1$ $(c = 1)$ and $\rho = -1$ $(c = -1)$. Next write $h(\varphi) = b(c-\cos\varphi)$, which vanishes iff $c = \cos\varphi$; because $|\cos\varphi| \leq 1$ and $|c| \geq 1$, the only roots of $h(\varphi)$ are found when $\rho = 1$ $(c = 1)$ which gives $\varphi = 0$ and $\rho = -1$ $(c = -1)$ which gives $\varphi = \pi$. □

Proof of Proposition 5.2. Apply eq. (4.3.8) page 160 in Fuller (1996) with $\omega = \lambda$, $p = 2n$, with n of the roots m_j equal to w_u and n roots equal to w_u^*; this gives

$$f_y(\lambda) = \frac{\sigma_\zeta^2}{2\pi}\left(\frac{1}{1-2w_u\cos\lambda + w_u^2}\right)^n \left(\frac{1}{1-2w_u^*\cos\lambda + w_u^{*2}}\right)^n. \tag{A.1}$$

Next recall the indentity

$$1 - 2a\cos\xi + w_u^2 = \left(1-ae^{i\xi}\right)\left(1-ae^{-i\xi}\right), \tag{A.2}$$

which, applied to from left to right to $c(\lambda, w_u) := 1 - 2w_u\cos\lambda + w_u^2$ gives $c(\lambda, w_u) = \left(1-w_ue^{i\lambda}\right)\left(1-w_ue^{-i\lambda}\right)$. Next, substituting $w_u = \rho_u e^{i\lambda_u}$ and setting $\varphi_u := \lambda+\lambda_u$, $\theta_u := \lambda - \lambda_u$, one finds $c(\lambda, w_u) = \left(1-\rho_ue^{i\varphi_u}\right)\left(1-\rho_ue^{-i\theta_u}\right)$. Because $w_u^* = \rho_u e^{-i\lambda_u}$, the same procedure applied to $c(\lambda, w_u^*)$ gives $c(\lambda, w_u^*) = \left(1-\rho_ue^{-i\varphi_u}\right)\left(1-\rho_ue^{i\theta_u}\right)$. Hence, using (A.2) again for $\xi = \varphi_u, \theta_u$, one can rewrite $c(\lambda, w_u)\,c(\lambda, w_u^*)$ in the denominator of (A.1) as

$$\begin{aligned} c(\lambda, w_u)\,c(\lambda, w_u^*) &= \left(1-\rho_ue^{i\varphi_u}\right)\left(1-\rho_ue^{-i\theta_u}\right)\left(1-\rho_ue^{-i\varphi_u}\right)\left(1-\rho_ue^{i\theta_u}\right) \\ &= \left(1-2\rho_u\cos\varphi_u + w_u^2\right)\left(1-2\rho_u\cos\theta_u + w_u^2\right), \end{aligned}$$

which gives (5.4) by substituting back $\varphi_u := \lambda + \lambda_u$, $\theta_u := \lambda - \lambda_u$.

In order to study $f_y(\lambda)$, we define the function $g(\lambda) := h(\lambda-\lambda_u, \rho_u)\,h(\lambda+\lambda_u,\rho_u)$ with $h(\varphi,\rho_u) := a_u - b_u\cos\varphi$, $a_u := 1+\rho_u^2$, $b_u := 2\rho_u$ and note that $f_y(\lambda)$ has a min (max) when $g(\lambda)$ has a max (min). Moreover $f_y(\lambda)$ is bounded when $h(\lambda-\lambda_u,\rho_u) > 0$ and $h(\lambda+\lambda_u,\rho_u) > 0$, which is always the case for $|\rho_u| < 1$ by Lemma A.2.

Define the notation $\dot g(\lambda) := \mathrm{d}g(\lambda)/\mathrm{d}\lambda$, $\ddot g(\lambda) := \mathrm{d}^2 g(\lambda)/\mathrm{d}^2\lambda$; one finds

$$\begin{aligned} \dot g(\lambda) &= -2b_u\sin\lambda\left(b_u\cos\lambda - a_u\cos\lambda_u\right), \\ \ddot g(\lambda) &= -2b_u\cos\lambda\left(b_u\cos\lambda - a_u\cos\lambda_u\right) + 2b_u^2\sin^2\lambda, \end{aligned}$$

where we have used standard trigonometric identities. Hence $\dot g(\lambda) = 0$ if and only if either $\sin\lambda = 0$ (i.e. $\lambda = 0, \pi$) or $\cos\lambda = c_u\cos\lambda_u$, where $c_u := \frac{a_u}{b_u} = \frac{1+\rho_u^2}{2\rho_u} > 0$ and $|c_u| \geq 1$, see proof of Lemma A.2. A solution to $\cos\lambda = c_u\cos\lambda_u$ exists iff $|c_u\cos\lambda_u| \leq 1$; in this case $\lambda = \arccos(c_u\cos\lambda_u)$ It is simple to note that $-1 \leq c_u\cos\lambda_u \leq 1 \iff -c_u^{-1} \leq \cos\lambda_u \leq c_u^{-1} \iff \arccos(c_u^{-1}) \leq \lambda_u \leq \arccos(-c_u^{-1})$, which gives (5.5). When $|c_u\cos\lambda_u| > 1$, there is no solution in λ of $\cos\lambda = c_u\cos\lambda_u$. Hence the stationary points of g and f are found at $\lambda = 0, \mp\pi$ and, when $|c_u\cos\lambda_u| \leq 1$, at $\lambda = \arccos(c_u\cos\lambda_u)$.

We next discuss signs of second derivatives. Assume first $\lambda_u < \arccos(c_u^{-1})$, so that $c_u\cos\lambda_u > 1$; one has $\ddot g(0) = -2b_u^2(1-c_u\cos\lambda_u) > 0$, and f_y has a max at $\lambda = 0$. Moreover $\ddot g(\mp\pi) = -2b_u^2(1+c_u\cos\lambda_u) < 0$, and f_y has a min at $\lambda = \mp\pi$. Next assume , i.e. $c_u\cos\lambda_u < -1$; one has $\ddot g(0) = -2b_u^2(1-c_u\cos\lambda_u) < 0$, and f_y has a min at $\lambda = 0$. Moreover $\ddot g(\mp\pi) =$

$-2b_u^2\,(1+c_u\cos\lambda_u) > 0$, and f_y has a max at $\lambda = \mp\pi$. Next if (5.5) holds, then $-1 \leq c_u \cos\lambda_u \leq 1$ and $\ddot{g}(0) = -2b_u^2\,(1-c_u\cos\lambda_u) < 0$, and $\ddot{g}(\mp\pi) = -2b_u^2\,(1+c_u\cos\lambda_u) < 0$, and f_y has a min at $\lambda = 0, \mp\pi$. Finally at $\lambda_\diamondsuit = \arccos(c_u\cos\lambda_u)$ one finds $\ddot{g}(\lambda_\diamondsuit) = 2b_u^2\sin^2\lambda_0 > 0$, so that f_y has a max at $\lambda = \mp\lambda_\diamondsuit$. □

Proof of Theorem 6.8. First we observe that by Lemma 3.2 and the polynomial rank factorization of $G(z)$ at z_u, one has

$$B_{u,0} = G(z_u) = -\xi_{u,0}\eta'_{u,0} = -(\operatorname{Re}\xi_{u,0} : \operatorname{Im}\xi_{u,0})\begin{pmatrix}\operatorname{Re}\eta'_{u,0}\\ \operatorname{Im}\eta'_{u,0}\end{pmatrix} + i(\operatorname{Re}\xi_{u,0} : -\operatorname{Im}\xi_{u,0})\begin{pmatrix}\operatorname{Im}\eta'_{u,0}\\ \operatorname{Re}\eta'_{u,0}\end{pmatrix},$$

where $\operatorname{Im} B_{u,0} \neq 0$ for $u : 0 < \lambda_u < \pi$ and $\operatorname{Im} B_{u,0} = 0$ for $u : \lambda_u \in \{0, \pi\}$; because

$$A_u(z) = (1-w_u^* z)h_u B_{u,0} + (1-w_u z)h_u^* B_{u,0}^* = (h_u B_{u,0} + h_u^* B_{u,0}^*) - (w_u^* h_u B_{u,0} + w_u h_u^* B_{u,0}^*)z,$$

one has

$$A_{u,0} = 2\operatorname{Re} h_u B_{u,0}, \qquad A_{u,1} = -2\operatorname{Re} w_u^* h_u B_{u,0}.$$

Because $h_u B_{u,0} = (\operatorname{Re} h_u \operatorname{Re} B_{u,0} - \operatorname{Im} h_u \operatorname{Im} B_{u,0}) + i(\operatorname{Re} h_u \operatorname{Im} B_{u,0} + \operatorname{Im} h_u \operatorname{Re} B_{u,0})$ one has

$$\begin{aligned}\operatorname{Re} h_u B_{u,0} &= -\operatorname{Re} h_u(\operatorname{Re}\xi_{u,0} : \operatorname{Im}\xi_{u,0})\begin{pmatrix}\operatorname{Re}\eta'_{u,0}\\ \operatorname{Im}\eta'_{u,0}\end{pmatrix} - \operatorname{Im} h_u(\operatorname{Re}\xi_{u,0} : -\operatorname{Im}\xi_{u,0})\begin{pmatrix}\operatorname{Im}\eta'_{u,0}\\ \operatorname{Re}\eta'_{u,0}\end{pmatrix},\\ \operatorname{Im} h_u B_{u,0} &= \operatorname{Re} h_u(\operatorname{Re}\xi_{u,0} : -\operatorname{Im}\xi_{u,0})\begin{pmatrix}\operatorname{Im}\eta'_{u,0}\\ \operatorname{Re}\eta'_{u,0}\end{pmatrix} + \operatorname{Im} h_u(\operatorname{Re}\xi_{u,0} : \operatorname{Im}\xi_{u,0})\begin{pmatrix}\operatorname{Re}\eta'_{u,0}\\ \operatorname{Im}\eta'_{u,0}\end{pmatrix}\end{aligned}$$

and because $w_u^* h_u B_{u,0} = (\operatorname{Re} w_u \operatorname{Re} h_u B_{u,0} + \operatorname{Im} w_u \operatorname{Im} h_u B_{u,0}) + i(\operatorname{Re} w_u \operatorname{Im} h_u B_{u,0} - \operatorname{Im} w_u \operatorname{Re} h_u B_{u,0})$ one finds

$$\operatorname{Re} w_u^* h_u B_{u,0} = \operatorname{Re} w_u \operatorname{Re} h_u B_{u,0} + \operatorname{Im} w_u \operatorname{Im} h_u B_{u,0}.$$

Hence $\gamma' A_u(z) = 0$ if and only if $\gamma \subset \operatorname{col}^\perp(\operatorname{col}\operatorname{Re}\xi_{u,0} \cap \operatorname{col}\operatorname{Im}\xi_{u,0})$ and for real roots $\gamma' B_{u,0} = 0$ if and only if $\gamma \subset \operatorname{col}^\perp \xi_{u,0}$. By the duality result in (6.4), one has $\operatorname{col}\xi_{u,0} = \operatorname{col}\beta_{u,1}$, which implies $\operatorname{col}\operatorname{Re}\xi_{u,0} = \operatorname{col}\operatorname{Re}\beta_{u,1}$, $\operatorname{col}\operatorname{Im}\xi_{u,0} = \operatorname{col}\operatorname{Im}\beta_{u,1}$. Hence $\operatorname{col}\gamma \subseteq \operatorname{col}(\operatorname{Re}\beta_{u,0} : \operatorname{Im}\beta_{u,0})$ and for real roots $\operatorname{col}\gamma \subseteq \operatorname{col}\beta_{u,0}$. □

Proof of Theorem 6.9. Because

$$\gamma'_u(z)G(z) = (z - z_u)^{m_u}\widetilde{\gamma}'_u(z)$$

where $\widetilde{\gamma}_u(z_u)$ has full column rank, see Franchi and Paruolo (2009) for the proof, one finds

$$\gamma'_u(z)\operatorname{inv}\Pi(z) = \frac{\gamma'_u(z)G(z)}{g(z)} = \frac{\widetilde{\gamma}'_u(z)}{g_u(z)},$$

so that $\gamma'_u(z)\operatorname{inv}\Pi(z)$ has no pole at $z = z_u$. □

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