UN-HIDING ROOTS IN VECTOR AUTOREGRESSIVE PROCESSES

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Abstract. As in Gregoir and Laroque (1993) we consider multivariate processes X_t such that, for some positive integer d, $\Delta^d X_t = C(L)\epsilon_t$ is $I(0)$ and, for some positive integer k, $det(C(z)) = (z-1)^k d(z)$, $d(1) \neq 0$; in addition, we assume that X_t is a vector autoregressive process, $\Pi(L)X_t = \epsilon_t$. First we show that $d = b - a$, where b and a are respectively the multiplicity of the unit root in $det(\Pi(z))$ and in $adj(\Pi(z))$, and $k = pd - b$, where p is the dimension of the process. Then we prove that the number m of polynomial cointegrating relations is equal to the order of integration d of the process and finally we use the algorithm in Franchi (2007b) to characterize the polynomial cointegrating vectors in terms of the autoregressive coefficients.

1. Introduction

2. Definitions

Definition 2.1 (Johansen (1996)). *The linear process $Y_t = C(L)\epsilon_t$, ϵ_t i.i.d.$(0,\Omega)$, is called $I(0)$ if $C(z) = \sum_{i=0}^{\infty} C_i z^i$ converges for $|z| < 1+\rho$ for some $\rho > 0$ and $C(1) \neq 0_p$; if $\Delta^d Y_t$ is $I(0)$ then Y_t is $I(d)$.*

Definition 2.2 (Gregoir and Laroque (1993)). *The process $Y_t = C(L)\epsilon_t$, ϵ_t i.i.d.$(0,\Omega)$, belongs to $\mathcal{C}$ if $C(z) = \sum_{i=0}^{\infty} C_i z^i$ converges for $|z| < 1+\rho$ for some $\rho > 0$, $C(1) \neq 0_p$ and*

$$det(C(z)) = (z-1)^k d(z)$$

where k is some positive integer and $d(z)$ is well defined and different from zero for $|z| < 1+\rho$ for some $\rho > 0$.

The notions of cointegration and polynomial cointegration are adaptations of the corresponding concepts for fractional processes (see Franchi, 2007b).

Definition 2.3. *The $I(d)$ process X_t is cointegrated if there exists γ_0 such that $\gamma_0' X_t$ is $I(d_0)$ with $0 \leq d_0 < d$. We call γ_0 a cointegrating vector.*

We say that a process X_t is cointegrated when it is integrated of a given order d and there exists a linear combination $\gamma_0' X_t$ of order $d_0 < d$.

Definition 2.4. *The $I(d)$ process X_t is polynomially cointegrated if there exist $\gamma(\Delta) := \gamma_0' + \sum_{i=1}^{n} \gamma_i' \Delta^i$, such that the transformed process $\gamma(\Delta) X_t$ is $I(d_n)$ with $0 \leq d_n < d_0$. We call $\gamma(\Delta)$ the polynomial cointegrating vector associated to γ_0 if there is no $\phi(\Delta) := \gamma_0' + \sum_{i=1}^{n_\phi} \phi_i' \Delta^i$ such that $\phi(\Delta) X_t$ is integrated of lower order than $\gamma(\Delta) X_t$. By cointegration structure we mean the collection of cointegrating and polynomial cointegrating vectors of X_t.*

Polynomial cointegration arises when the $I(d_0)$, $d_0 < d$, cointegrating relation $\gamma_0' X_t$ linearly combined with $\Delta^n X_t$, $n = d - d_0$, of the same order of integration results of lower order $d_n < d_0$.

That cointegration is a necessary condition for polynomial cointegration, it is immediately seen because if γ_0 is not a cointegrating vector, $\gamma_0' X_t \in I(d)$ implies $\gamma_0' X_t + \gamma_1' \Delta X_t + \cdots + \gamma_n' \Delta^n X_t \in I(d)$ for any n. For the same reason, $\sum_{i=0}^{j} \gamma_i' \Delta^i X_t \in I(d_j)$, $d_j < d$, is a necessary condition for $\sum_{i=0}^{j+1} \gamma_i' \Delta^i X_t \in I(d_{j+1})$, $d_{j+1} < d_j$. Thus, for a given cointegrating vector γ_0 we need to find the linear combination of levels and (possibly) differences which has the lowest possible order of integration and this is what we call the polynomial cofraction vector associated to γ_0. This rules out the existence of an additional transformation $\gamma_0' X_t + \sum_{i=1}^{n_\phi} \phi_i' \Delta^i X_t$ which has lower order of integration and thus guarantees the complete characterization of the cointegration structure.

3. The VAR process and the class $\mathcal{C}$

Consider the vector autoregressive process

$$\Pi(L) X_t = \epsilon_t \tag{3.1}$$

where X_t is $p \times 1$, $\Pi(L)$ is a finite matrix polynomial in the lag operator L and ϵ_t is independent and identically distributed with mean 0 and positive definite variance $\Omega > 0$ (denoted $i.i.d.(0, \Omega)$).

Under the assumption that the roots of $det(\Pi(z)) = 0$ are either at $z = 1$ or outside the unit circle and $\Pi(1) \neq 0_p$, it is well known (see Franchi, 2007a) that X_t in (3.1) is integrated of order $d = b - a > 0$, where $b > 0$ is found in

$$det(\Pi(z)) = (z-1)^b g(z)$$

and $0 \leq a < b$ in

$$adj(\Pi(z)) = (z-1)^a G(z).$$

The polynomial $g(z)$ is different from zero for $|z| < 1+\rho$ for some $\rho > 0$ and $G(z)$ is a $p \times p$ matrix polynomial that satisfies $rank(G(1)) \leq p-1$ and $G(1) \neq 0_p$. Then

$$\Pi(z)^{-1} = \frac{G(z)}{(z-1)^{b-a} g(z)}, \; z \neq \{1\} \cup \{z \in \mathbb{C} : g(z) = 0\}$$

and because $g(z) \neq 0$ for $|z| < 1 + \rho$ for some $\rho > 0$, the function

$$C(z) := \frac{G(z)}{g(z)}, \; z \neq \{z \in \mathbb{C} : g(z) = 0\},$$

converges on the same disc and it is such that $C(1) = \frac{G(1)}{g(1)} \neq 0_p$. This shows that $\Delta^{b-a} X_t$ is $I(0)$.

Theorem 3.1. *Let the roots of $det(\Pi(z)) = 0$ be either at $z = 1$ or outside the unit circle and $\Pi(1) \neq 0_p$; then $\Delta^d X_t$, $d = b - a$, where b and a are respectively the multiplicity of the unit root in $det(\Pi(z))$ and in $adj(\Pi(z))$, belongs to $\mathcal{C}$ with $k = pd - b$ and $d(z) = g(z)^{-1}$.*

Proof. The identity $\Pi(z) adj(\Pi(z)) = det(\Pi(z)) I_p$ gives

$$\Pi(z) G(z) = (z-1)^{(b-a)} g(z) I_p.$$

Then

$$det(\Pi(z)) det(G(z)) = (z-1)^{pd} g(z)^p$$

and $det(\Pi(z)) = (z-1)^b g(z)$ imply

$$det(G(z)) = (z-1)^{pd-b} g(z)^{p-1}.$$

Because $C(z) := \frac{G(z)}{g(z)}$ we have

$$det(C(z)) = g(z)^{-p} det(G(z)) = (z-1)^{pd-b} g(z)^{-1}$$

and $pd - b > 0$ follows from $1 \leq b \leq pd$ because the upper equality holds if and only if $\Pi(1) = 0_p$. Because $g(z)$ is different from zero for $|z| < 1 + \rho$ for some $\rho > 0$, it follows that $d(z) = g(z)^{-1}$ is well defined and different from zero on the same disc. This completes the proof that $\Delta^d X_t \in \mathcal{C}$. ■

Theorem 3.1 shows that if the characteristic polynomial of X_t in (3.1) has roots at $z = 1$ or outside the unit circle, "differenced enough" means taking $\Delta^d X_t$, where $d = b - a$, and then $\Delta^d X_t \in \mathcal{C}$ follows immediately. Moreover, k in $det(C(z)) = (z-1)^k d(z)$ is equal to the number of variables p times the order of integration d minus the multiplicity of the unit root in the characteristic polynomial b, $k = pd - b$, and $d(z)$ is equal to the inverse of $g(z)$, which is found by factoring out the unit root from $det(\Pi(z))$.

Note that processes with different orders of integration can share the same k and thus belong to the same class; indeed this is true by definition, because the processes in $\mathcal{C}$ have been transformed to be $I(0)$ by "proper differencing". As it will be clarified in the next section, this implies that the cointegration structure of the processes in the class $\mathcal{C}$ is undetermined.

4. Polynomial cointegration

The representation theory in Gregoir and Laroque (1993) is developed on the basis of the integral operator S and it is shown that if a process $Y_t \in \mathcal{C}$, then, for some positive integer m, there exists a matrix polynomial

$$\vartheta^h(u) := \sum_{k=1}^{h} \vartheta_k^h u^k, \ h = 1, \cdots, m,$$

of some dimension $r_h \times p$, such that the process

$$\vartheta^h(S)Y_t := \sum_{k=1}^{h} \vartheta_k^h S^i Y_t, \ h = 1, \cdots, m,$$

is $I(0)$ event though $S^i Y_t$ is $I(i)$, $i = 1, \cdots, m$.

Here we want to characterize the number m of polynomial cointegrating vectors $\vartheta^h(u)$ and their coefficients ϑ_k^h. Because $\Delta^d X_t \in \mathcal{C}$ and $S^k \Delta^d = \Delta^{d-k}$,

$$\vartheta^h(S)\Delta^d X_t = \Delta^{d-h} \sum_{k=1}^{h} \vartheta_k^h \Delta^{h-k} X_t, \ h = 1, \cdots, m,$$

shows that $\vartheta^h(S)\Delta^d X_t$ is $I(0)$ if and only if

$$\sum_{k=1}^{h} \vartheta_k^h \Delta^{h-k} X_t,\ h = 1, \cdots, m,$$

is $I(d-h)$. Then we can prove the following result.

Theorem 4.1. *Let the roots of det$(\Pi(z)) = 0$ be either at $z = 1$ or outside the unit circle, the coefficients α_s, β_s and $\theta_{s,k}$ be as defined in (A.3)-(A.6) in the Appendix and $\pi_s \neq 0$; then*

$$m = d,$$

where $d = b - a$ and b and a are respectively the multiplicity of the unit root in det$(\Pi(z))$ and in adj$(\Pi(z))$. The process $\vartheta^h(S)\Delta^d X_t$ is $I(0)$ if and only if, for $h = 1, \cdots, d$,

$$\vartheta_h^h = (\beta_{d-h+1}\varphi)'$$

and

$$\vartheta_{h-k}^h = (-1)^{k+1}(\bar{\alpha}_{d-h+1}\varphi)'\theta_{d-h+1,k},\ k = 1, \cdots, h,$$

for some $\varphi \neq 0_{r_s}$.

Proof. By Theorem A.2 in the Appendix

$$\phi_s(\Delta)X_t := \sum_{k=0}^{n_s} \phi_{s,k}\Delta^k X_t,\ s = 1, \cdots, d,$$

of dimension $r_s \times p$, is $I(d_{\phi_s})$, $d_{\phi_s} \geq s-1$, and the equality holds if and only if $\phi_s(\Delta) = \varphi' p_s(\Delta)$ for some $\varphi \neq 0_{r_s}$. Then

$$\sum_{k=1}^{h} \vartheta_k^h \Delta^{h-k} X_t,\ h = 1, \cdots, m,$$

is $I(d-h)$ if and only if for $s = d - h + 1$ we have

$$\sum_{k=0}^{h-1} \vartheta_{h-k}^h (1-z)^k = \varphi' p_s(1-z),\ h = 1, \cdots, m.$$

Then

$$p_s(1-z) = \sum_{k=0}^{d-s} p_k^s (1-z)^k,\ s = 1, \cdots, d,$$

where $p_0^s := \beta_s'$ and $p_k^s := (-1)^{k+1}\bar{\alpha}_s'\theta_{s,k}$, gives

$$p_{d-h+1}(1-z) := \sum_{k=0}^{h-1} p_k^{d-h+1}(1-z)^k,\ h = 1, \cdots, d.$$

This shows $m = d$ and

$$\vartheta^h_{h-k} = \varphi' p_k^{d-h+1}$$

for some $\varphi \neq 0_{r_s}$ and completes the proof. ■

Theorem 4.1 proves that the number of polynomial cointegrating relations is equal to the order of integration of X_t; then it shows how to construct the polynomial cointegrating vectors $\vartheta^h(u)$ from the coefficients α_s, β_s and $\theta_{s,k}$ as defined by the algorithm in the Appendix. These coefficients are found by exploiting the reduced rank restrictions that are satisfied by the coefficients of the matrix polynomial $\Pi(z)$ when its inverse has a pole of order d at $z = 1$.

Note that for processes of dimension p and different orders of integration $d = 1, \cdots$ we can always find b such that $k = pd - b$ is constant; this means that if we construct $\mathcal{C}$ on the basis of k in $det(C(z)) = (z-1)^k d(z)$, because $m = d$ we end up with by collecting together processes with different cointegration structures. Thus the cointegration structure of the class $\mathcal{C}$ is undetermined.

5. A different class of interest

It is appealing to define the class of interest in such a way that the cointegration structure is invariant; in this way we know exactly what are the polynomial cointegrating relations and the corresponding error correction representation of the members of this class.

Because $G(z)$ is a $p \times p$ matrix polynomial that satisfies $rank(G(1)) \leq p-1$, it follows that

$$adj(G(z)) = (z-1)^{k_a} H(z),\ H(1) \neq 0_p,\ k_a \geq 0,$$

and $H(z)$ is a matrix polynomial of finite degree. The reason is that when $rank(G(1)) < p-1$, $adj(G(1)) = 0_p$ and thus each entry of $adj(G(z))$ contains the factor $z-1$ and when $rank(G(1)) = p-1$, $adj(G(1)) \neq 0_p$ and thus $k_a = 0$.

In the next Proposition we find that k_a is equal to the number of variables minus one times the order of integration minus the multiplicity of the unit root in the characteristic polynomial.

Proposition 5.1. *The multiplicity of the unit root in $adj(G(z))$ is*

$$k_a = (p-1)d - b.$$

Proof. Using

$$det(G(z)) = (z-1)^k g(z)^{p-1},\ g(1) \neq 0,$$

where $k = pd - b > 0$, we write the inverse of $G(z)$ as

$$G(z)^{-1} = \frac{H(z)}{(z-1)^{k-k_a} g(z)^{p-1}},\ z \neq \{1\} \cup \{z \in \mathbb{C} : g(z) = 0\}.$$

Because $det(G(1)) = 0$, $H(1) \neq 0_p$ and $g(1) \neq 0$, this shows that $G(z)^{-1}$ has a pole of order $k - k_a > 0$ at the unit root. From

$$\Pi(z)^{-1} = \frac{G(z)}{(z-1)^{b-a} g(z)},\ z \neq \{1\} \cup \{z \in \mathbb{C} : g(z) = 0\}$$

we get

$$\Pi(z) = (z-1)^d g(z) G(z)^{-1},\ 0 < |z-1| < \rho,$$

and write

$$\Pi(z) = \frac{(z-1)^d}{(z-1)^{k-k_a}} \frac{H(z)}{g(z)^{p-2}}.$$

Then $\Pi(z)$ is well defined and different from 0_p at $z = 1$ if and only if

$$k - k_a = d$$

and

$$k_a = (p-1)d - b$$

immediately follows. ■

Proposition 5.1 shows the interesting result that the difference between the multiplicity of the unit root in $det(C(z))$ and in $adj(C(z))$ is equal to the order of integration of the process, $k - k_a = d$. Exactly as it is for autoregressive processes, where it is not interesting to collect processes on the basis of the multiplicity of the unit root in $det(\Pi(z))$, when we start from the moving average representation we need to focus on the difference between the multiplicity in $det(C(z))$ and in $adj(C(z))$ and not on k in $det(C(z))$ alone. It is only by constructing the class by collecting processes that share the same value for $k - k_a$, that is the same order of integration, that the cointegration structure of the members of the class is homogeneous and thus uniquely determined.

Then we define the class of interest in the following way.

Definition 5.2. *The process $Y_t = C(L)\epsilon_t$, ϵ_t i.i.d.$(0, \Omega)$, belongs to $\mathcal{C}_m$ if $C(z) = \sum_{i=0}^{\infty} C_i z^i$ converges for $|z| < 1 + \rho$ for some $\rho > 0$, $C(1) \neq 0_p$,*

$$det(C(z)) = (z-1)^k d(z)$$

where k is some positive integer and $d(z)$ is well defined and different from zero for $|z| < 1 + \rho$ for some $\rho > 0$,

$$adj(C(z)) = (z-1)^{k_a} D(z),$$

where $0 \leq k_a < k$ and $D(1) \neq 0_p$, and

$$m = k - k_a.$$

In this way the class $\mathcal{C}_1$ collects the processes that are $I(1)$ and such that $\vartheta_1^1 X_t$ is $I(0)$, $\mathcal{C}_2$ the processes that are $I(2)$ and such that $\vartheta_1^1 \Delta X_t$ and $\vartheta_1^2 \Delta X_t + \vartheta_2^2 X_t$ are $I(0)$, $\mathcal{C}_3$ the processes that are $I(3)$ and such that $\vartheta_1^1 \Delta^2 X_t$, $\vartheta_1^2 \Delta^2 X_t + \vartheta_2^2 \Delta X_t$ and $\vartheta_1^3 \Delta^2 X_t + \vartheta_2^3 \Delta X_t + \vartheta_3^3 X_t$ are $I(0)$, and so on.

If X_t is an autoregressive process, one can use Theorem 4.1 to characterize the cointegrating vectors in terms of the autoregressive coefficients. It is evident that the analysis in Gregoir (1999) suffers from the same problem.

6. Conclusion

We have studied

Appendix A

Here we describe the recursive algorithm developed in Franchi (2007b) for the characterization of the restrictions on the coefficients of a matrix polynomial $\Pi(z)$ so that its inverse has a pole of order d at $z = 1$. The proofs and more detailed explanations of the results are found there.

The algorithm defines d reduced rank matrices π_{s-1}, $s = 1, \cdots, d$, that are used to construct the cointegrating vector β_s and the corresponding adjustment coefficient α_s and stops at step $s = d + 1$ with a full rank matrix π_d which corresponds to the rank condition in the spirit of Johansen (1996). In addition, the algorithm reveals the restrictions that are responsible for the existence of polynomial cointegration for all but one cofraction vector.

The following notation is needed; whenever we have a $p \times p$ reduced rank matrix π of rank r, say, and we write it as $\pi = -\xi\eta'$ it is understood that ξ and η are full rank matrices of dimension $p \times r$. The matrices ξ and η are not unique but the conclusions do not depend on the choice made. For any $p \times r$ matrix γ of rank $r \leq p$, we let $\gamma_\perp$ of dimension $p \times p - r$ be a basis of the orthogonal complement of $sp(\gamma)$, so that $\gamma'\gamma_\perp = 0$ and $\gamma_\perp'\gamma = 0$; in particular we let $\gamma_\perp = 0$ if $r = p$ and $\gamma_\perp = I_p$ if $r = 0$. Furthermore we let $\bar{\gamma} := \gamma(\gamma'\gamma)^{-1}$ and denote $P_\gamma := \bar{\gamma}\gamma' = \gamma\bar{\gamma}'$ the projection matrix onto $sp(\gamma)$. We use the notation $\Pi_n := \frac{1}{n!}\left(\frac{d^n}{dz^n}\Pi(z)\right)\big|_{z=1}$, $G_n := \frac{1}{n!}\left(\frac{d^n}{dz^n}G(z)\right)\big|_{z=1}$ and $g_n := \frac{1}{n!}\left(\frac{d^n}{dz^n}g(z)\right)\big|_{z=1}$ for $n = 0, \cdots$.

The algorithm

The identity $\Pi(z)adj(\Pi(z)) = adj(\Pi(z))\Pi(z) = det(\Pi(z))I_p$ gives

$$\Pi(z)G(z) = G(z)\Pi(z) = (z-1)^d g(z) I_p.$$

For $N = d + d_g$, where d_g is the degree of $g(z)$, we write

$$\Pi(z)G(z) = \sum_{n=0}^{N} a_n (z-1)^n \text{ and } G(z)\Pi(z) = \sum_{n=0}^{N} b_n (z-1)^n$$

where

$$a_n := \sum_{k=0}^{n} \Pi_k G_{n-k} \text{ and } b_n := \sum_{k=0}^{n} G_k \Pi_{n-k}.$$

Then

$$a_n = b_n = 0_p,\ n = 0, \cdots, d-1, \tag{A.1}$$

and

$$a_{d+n} = b_{d+n} \neq 0_p,\ n = 0, \cdots, d_g, \tag{A.2}$$

follow immediately. The algorithm uses (A.1) to find out the sequence of reduced rank matrices π_s and the coefficients α_s, β_s and $\theta_{s,k}$ and (A.2) to characterize the cointegration structure of the model.

The coefficients are defined recursively as follows: let $r_0 = 0$, $\alpha_0 = \beta_0 = 0_p$, $\delta_{0\perp} = \zeta_{0\perp} = I_p$ and define

$$\theta_{0,k} := \Pi_{k-1},\ k = 1, \cdots, d+1,$$

and

$$\pi_0 := \delta'_{0\perp}\theta_{0,1}\zeta_{0\perp}$$

of dimension $p \times p$. Because $\Pi(z)$ has a unit root, π_0 has reduced rank and we write

$$\pi_0 = -\xi_1\eta'_1$$

where ξ_1 and η_1 are full rank matrices of dimension $p \times r_1$; then we define $\delta_{1\perp} := \delta_{0\perp}\xi_{1\perp}$, $\zeta_{1\perp} := \zeta_{0\perp}\eta_{1\perp}$ of dimension $p \times (p - r_1)$ and $\alpha_1 = \bar{\delta}_{0\perp}\xi_1$, $\beta_1 = \bar{\zeta}_{0\perp}\eta_1$ of dimension $p \times r_1$.

Given

$$\alpha_s,\ \beta_s,\ \delta_{s\perp},\ \zeta_{s\perp},\ \text{and } \theta_{s-1,k}$$

for $s = 1, \cdots, n$ and $k = 1, \cdots, d - s + 2$, we define

$$\theta_{n,k} := \theta_{n-1,1}\sum_{i=0}^{n-1}\bar{\beta}_i\bar{\alpha}'_i\theta_{i,k} + \theta_{n-1,k+1},\ k = 1, \cdots, d-n+1, \tag{A.3}$$

of dimension $p \times p$ and

$$\pi_n := \delta'_{n\perp}\theta_{n,1}\zeta_{n\perp} \tag{A.4}$$

of dimension $(p - \sum_{i=1}^{n} r_i) \times (p - \sum_{i=1}^{n} r_i)$. If π_n has full rank the algorithm stops. If π_n has reduced rank we write

$$\pi_n = -\xi_{n+1}\eta'_{n+1}$$

where ξ_{n+1} and η_{n+1} are full rank matrices of dimension $(p - \sum_{s=1}^{n} r_s) \times r_{n+1}$ and define

$$\alpha_{n+1} := \bar{\delta}_{n\perp}\xi_{n+1},\ \beta_{n+1} := \bar{\zeta}_{n\perp}\eta_{n+1} \tag{A.5}$$

of dimension $p \times r_{n+1}$,

$$\delta_{n+1\perp} := \delta_{n\perp}\xi_{n+1\perp}\ \zeta_{n+1\perp} := \zeta_{n\perp}\eta_{n+1\perp} \tag{A.6}$$

of dimension $p \times (p - \sum_{i=1}^{n+1} r_i)$; then we compute (A.3) and (A.4) for $n+1$ and iterate.

We can prove the following the result.

Theorem A.1. *Let the coefficients be as defined in (A.3)-(A.6); then, for $n = 1, \cdots, d-1$, π_n has reduced rank and for $n = d$, π_d has full rank.*

Proof. See Franchi (2007b). ■

Theorem A.1 finds the d reduced rank restrictions that are satisfied by the coefficients of a matrix polynomial $\Pi(z)$ whose inverse has a pole of order d at $z = 1$. We show that for $s = 1, \cdots, d$, π_{s-1} in (A.4) has reduced rank r_s and defines α_s and β_s of dimension $p \times r_s$ in (A.5); then $\theta_{s,k}$, $k = 1, \cdots, d-s+1$, of dimension $p \times p$ in (A.3) can be computed by using the previously defined coefficients.

As stated in Theorem A.2 below, these coefficients characterize the cointegration structure of X_t in (3.1).

Theorem A.2. *Let the roots of $det(\Pi(z)) = 0$ be either at $z = 1$ or outside the unit circle, the coefficients be as in (A.3)-(A.6) in the Appendix, and define*

$$p_s(1-z) := \beta_s' - \bar{\alpha}_s' \sum_{k=1}^{d-s} (-1)^k \theta_{s,k} (1-z)^k, \ s = 1, \cdots, d-1,$$

of dimension $r_s \times p$ and

$$p_d(1-z) := \beta_d'$$

of dimension $r_d \times p$. Then the process

$$p_s(\Delta) X_t, \ s = 1, \cdots, d,$$

is $I(s-1)$. Moreover

$$\phi_s(\Delta) X_t, \ s = 1, \cdots, d,$$

of dimension $r_s \times p$, is $I(d_{\phi_s})$, $d_{\phi_s} \geq s-1$, and the equality holds if and only if $\phi_s(\Delta) = \varphi' p_s(\Delta)$ for some $\varphi \neq 0_{r_s}$.

Proof. See Franchi (2007b). ■

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