ENERGY TRANSFER IN NONLINEAR MECHANICAL OSCILLATORS: AN ANALYTICAL STUDY OF THE CONTINUATION OF THE PERIODIC ORBITS OF THE UNDERLYING HAMILTONIAN SYSTEM

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This work is part of a more general work concerning the energy transfer and dissipation in a system of nonlinear coupled mechanical oscillators. We intend to study the transfer of oscillations from the main body of the mechanical system to a nonlinear adjunction and their final decrease.

By energy transfer we refer to the controlled spatial transfer of vibrational energy from the point of its initial generation to a different point where it eventually localizes. The energy transfer mechanism here is realized through resonance capture.

It has been shown through simulations and experimental demonstrations [1-6] that the energy transfer from the initial point to the nonlinear energy sink occurs through the fundamental (1:1) resonance, as well as, subharmonic (m:n) resonances.

In this first step, we focus only on the undamped system, that plays an essential role in energy transfer phenomena in the damped system [1-6], studying the continuation of the periodic orbits of the perturbed Hamiltonian system and their stability with the use of theorems of Poincare [7] and Meletlidou-Stagika [8]. The stable periodic orbits that are continued form the core of the subharmonic resonances, and with this work we hope to contribute to the study of the role that these resonances play in the energy transfer mechanism.

We study a system of two degrees of freedom coupled oscillators with essential nonlinearities and weak viscous damping

$$\begin{aligned} &\ddot{x}+\varepsilon\lambda\dot{x}+Cx^3+\varepsilon(x-y)=0,\\ &\ddot{y}+\varepsilon\lambda\dot{y}+\omega_2^2 y+\varepsilon(y-x)=0, \end{aligned} \tag{1}$$

which, for $\lambda=0,$ becomes a Hamiltonian system with Hamiltonian

$$H=\frac{P_x^2}{2}+\frac{P_y^2}{2}+\frac{C}{4}x^4+\frac{\varepsilon}{2}\left(x^2+y^2\right)+\frac{\omega^2}{2}y^2-\varepsilon xy. \tag{2}$$

Poincare proved that for a perturbed Hamiltonian system of n degrees of freedom $H = H_0 + \varepsilon H_1$ where H_0 is integrable, has bounded orbits and is isoenergetically non degenerate, if, for a T-periodic orbit on the resonant torus of the Hamiltonian H_0, the vector $\frac{\partial\langle H_1\rangle}{\partial\varphi}$ has a simple zero (φ_i^*, J_i^*) and $|\frac{\langle\partial^2 H_1\rangle}{\partial\varphi_i\varphi_j}| \neq 0, (i, j = 1, 2, \ldots, n-1)$ then for an open interval of values of ε around zero, the perturbed Hamiltonian $H = H_0 + \varepsilon H_1$ has one T-periodic orbit that depends analytically on ε and for $\varepsilon = 0$ it coincides with the unperturbed periodic orbit. In the case where the Hamiltonian H_0 is degenerate, we use the Hamiltonian $\exp(H)$. The two systems have the same orbits, written on a different time unit, and the same action- angle variables. If H_0 is isoenergetically non-degenerate then $\exp(H_0)$ is non-degenerate. Hence, if we work on the new system, which is non-degenerate, we can transfer our conclusions to the initial Hamiltonian.

The linear stability of the periodic orbit of the perturbed system predicted by the above theorem for the case of the two degrees of freedom Hamiltonian system H is determined by the Floquet characteristic exponents $\sigma = \sqrt{\varepsilon}\sigma_1 + O(\varepsilon)$ [9]. If σ_1 is imaginary then the orbit is stable. If it is real then the orbit is unstable.

Meletlidou and Stagika proved a criterion for the continuation of the non-isolated periodic orbits for which the average value $\langle H_1 \rangle$ is constant along the periodic orbits of the resonant torus and Poincare's theorem is not applicable. Particularly, they proved that if

$$\left|\frac{\partial^2 H_0}{\partial \mathbf{J}^2}\right| \neq 0, \qquad rank\left|\frac{\partial \mathbf{F}_2}{\partial \varphi}\right| = n-1$$

where

$$\mathbf{F}_2(\varphi, \mathbf{J}) = \langle \frac{\partial^2 H_1}{\partial \mathbf{w}^2} ?_0 (w^{(1)} - w^{(1)}(0)) + \frac{\partial^2 H_1}{\partial \mathbf{w} \partial \mathbf{J}} |_0 \, \mathbf{J}^{(1)} \rangle$$

then these periodic orbits are continued for sufficiently small $\varepsilon \neq 0$ with the same period T. The importance of the above theorems is that they allow us to determine analytically up to $O(\varepsilon)$ the initial values of the periodic orbits of H_0 that are continued to the perturbed Hamiltonian. We apply the above theorems to Hamiltonian (2) and find the continued periodic orbits. For the periodic orbits that are proved to be continued by Poincare's theorem we determine their stability. We also take the Poincare section of Hamiltonian (2) and make the comparison between our analytical and numerical results.

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