The scope of logic: deduction, abduction, analogy

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1. Natural logic

The present form of mathematical logic originated in the twenties and early thirties from the partial merging of two different traditions, the algebra of logic and the logicist tradition (see [27], [41]). This resulted in a new form of logic in which several features of the two earlier traditions coexist. Clearly neither the algebra of logic nor the logicist's logic is identical to the present form of mathematical logic, yet some of their basic ideas can be distinctly recognized within it.

One of such ideas is Boole's view that logic is the study of the laws of thought. This is not to be meant in a psychologistic way. Frege himself states that the task of logic can be represented "as the investigation of *the* mind; [though] of *the* mind, not of minds" [17, p. 369]. Moreover Frege never charges Boole with being psychologistic and in a letter to Peano even distinguishes between the followers of Boole and "the psychological logicians" [16, p. 108]. In fact for Boole the laws of thought which are the object of logic belong "to the domain of what is termed *necessary* truth" [2, p. 404]. For him logic does not depend on psychology, on the contrary psychology depends on logic insofar as it is only through an investigation of logical operations that we could obtain "some probable intimations concerning the nature and constitution of the human mind" [2, p. 1]. Logic is normative, not descriptive. For, the laws of thought do not "manifest their presence otherwise than by merely prescribing the conditions of formal inference" [2, p. 419]. They are, "properly speaking, the laws of *right* reasoning only" [2, p. 408]. So they "form but a *part* of the system of laws by which the actual processes of reasoning, whether right or wrong, are governed" [2, p. 409].

Boole's idea that logic is the study of the laws of thought was taken over by Hilbert. According to him logic is "a discipline which expresses the structure of all our thought" [31, p. 125]. This is possible because our understanding "does not exercise any mysterious art, on the contrary, it proceeds only according to certain well-defined rules" [29, p. 9]. Such rules "form a closed system that can be discovered and definitively stated". They are captured by the formal systems of proof theory, where the "formula game is carried out according to certain definite rules in which the *technique of our thinking* is expressed". The fundamental idea of proof theory is "none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds". Making such a protocol is possible because "thinking, it so happens, parallels speaking and writing: we form statements and place them one behind another" [30, p. 475]. In the formal systems of proof theory each symbol denotes "an object of our thought" [28, p. 131]. Moreover, "the axioms and provable propositions, that is, the formulas resulting from this procedure, are copies of the thoughts constituting customary mathematics as it has developed till now" [30, p. 465].

Hilbert assumes that there is a strict correspondence between our thinking and the formal systems of proof theory. This agrees with a logical tradition where Leibniz maintains that what he calls 'mathematical logic' is "an art of forming and ordering characters in such a way that they report thoughts and are to each other in the same relation in which thoughts are to each other" [39, p. 80]. Thoughts are represented by formal expressions in such a way that, "if the idea of the thing to be expressed consists of the ideas of certain things, then the expression of that thing consists of the characters for those specific things" [39, pp. 80-81]. Then, "since the analysis of concepts thus corresponds exactly to the analysis of a character, we need merely to see the characters in order to have adequate notions brought to our mind freely and without effort" [38, p. 193]. Not only the logical calculus is capable of representing reasoning faithfully but even directs it infallibly because through it "writing and thinking will keep pace with each other or, rather, writing will be the thread of thinking" [36, VII, p. 14].

Gentzen's calculus of natural deduction can be viewed as a further development of Boole's idea that logic is the study of the laws of thought. According to Gentzen, contrary to Hilbert's claim the formalization of logical deduction developed by Frege, Russell, and Hilbert is "rather far removed from the forms of deduction used in practice in mathematical proofs". One should "set up a formal system which comes as close as possible to actual reasoning" [18, p. 68]. This is obtained by means of the calculus of natural deduction which "reflects as accurately as possible the actual logical reasoning involved in mathematical proofs" 3, p. 74]. The calculus has "a close affinity to actual reasoning" [18, p. 80]. Gentzen's cut elimination theorem can be viewed as stating that in actual reasoning we will not lose by using only direct reasoning, i.e. reasoning which "is not roundabout". In a cut free proof no concepts enter "other than those contained in its final result, and their use was therefore essential to the achievement of that result" [18, p. 69].

Within the algebra of logic Boole's idea that logic is the study of the laws of thought was further developed by Peirce. According to him "in logic our great object is to analyse all the operations of reason and reduce them to their ultimate elements". This is the primary task of logic, while "to make a calculus of reasoning is a subsidiary object" [43, 3.173, footnote 2]. The operations of a calculus of reasoning should closely correspond to the operations of reason. So a "system devised for the investigation of logic should be as analytical as possible, breaking up inferences into the greatest possible number of steps" [43, 4.373]. The system should "enable us to separate reasoning into its smallest steps so that each one may be examined by itself" [44, III, 405].

Peirce's view was taken over by Gentzen and more explicitly by Prawitz. In Prawitz's view, the calculus of natural deduction "is an attempt to isolate the essential deductive operations and to break them down as far as possible" [47, p. 244]. A proof "built up from Gentzen's atomic inferences is *completely analysed* in the sense that one can hardly imagine the possibility of breaking down his atomic inferences into some simpler inferences". Admittedly, the claim that through Gentzen's atomic inferences the essential deductive operations have been isolated "is not to be understood as a claim that these operations mirror all informal deductive practices". For, this "would be an unreasonable demand in view of the fact that informal practices may sometimes contain logically insignificant irregularities" [47, p. 245]. Nevertheless, it can be claimed that Gentzen's atomic inferences "correspond closely to procedures common in intuitive reasoning". When informal proofs such as are encountered in mathematics are formalized in terms of them, "the main structure of the informal proofs can often be preserved". In particular, Gentzen's atomic inferences are "closely related to the interpretation of the logical signs" [46, p. 7]. Indeed, the essential logical content of intuitive logical operations "can be understood as composed of the atomic inferences isolated by Gentzen. It is in this sense that we may understand the terminology *natural* deduction" [47, p. 245].

Therefore "Gentzen's systems of natural deduction are not arbitrary formalizations of first order logic but constitute a significant analysis of the proofs in this logic" [47, p. 246]. It seems "fair to say that no other system is more convincing in this respect" [47, p.

246, footnote 1]. Prawitz's normalization theorem can be viewed as stating that in actual reasoning we will not lose by using only direct reasoning, i.e. reasoning which "proceeds from the assumption to the conclusion by first only using the meaning of the assumptions by breaking them down in their components (the analytical part), and then only verifying the meaning of the conclusions by building them up from their components (the synthetical part)" [47, p. 258].

2. Gödel's intervention

The connection between the views of Hilbert, Gentzen and Prawitz and Boole's idea that logic is the study of the laws of thought seems to me quite clear. But if we accept it, then we must ask: Does mathematical logic provide an adequate implementation of Boole's idea?

Now, its implementation reduces mathematical reasoning to deduction, specifically, to deduction from given axioms. For example Shoenfield states that "mathematical logic is the study of the type of reasoning done by mathematicians" [51, p. 1]. We may "describe what a mathematician does as follows. He presents us with certain basic concepts and certain axioms about these concepts". Then he "proceeds to define derived concepts and to prove theorems about both basic and derived concepts. The entire edifice which he constructs, consisting of basic concepts, derived concepts, axioms and theorems, is called an *axiom system*" [51, p. 2]. So mathematical reasoning consists of deduction in some axiomatic system.

Given such an explanation my previous question amounts to: Does deduction provide an adequate account of mathematical reasoning? In my opinion no and Gödel's first incompleteness theorem provides evidence for that. This, however, is not the standard view. What can be considered as the standard view is expressed by Turing as follows. Mathematical reasoning is "the exercise of a combination of two faculties, which we may call *intuition* and *ingenuity*. The activity of the intuition consists in making spontaneous judgments which are not the result of conscious trains of reasoning" [58, pp. 208-9]. The necessity for using the intuition is "greatly reduced by setting down formal rules for carrying out inferences". Indeed, in pre-Gödel times it was thought that "all the intuitive judgments of mathematics could be replaced by a finite number of these rules. The necessity for intuition would then be entirely eliminated". Only ingenuity would be required to "determine which steps are the most profitable for the purpose of proving a particular proposition". For, "we are always able to obtain from the rules of formal logic a method of enumerating the propositions proved by its means". Specifically, we can "imagine that all proofs take the form of a search through this enumeration for the theorem for which a proof is desired. In this way ingenuity is replaced by patience" [58, p. 209]. After Gödel, however, we must recognize that it is impossible to find "a formal logic which wholly eliminates the necessity of using intuition" [58, p. 210].

So, in Turing's view, Gödel's result does not compel us to give up the idea that deduction provides an adequate account of mathematical reasoning. It only forces us to acknowledge that intuition plays an essential role in it, though a role which can be limited to a minimum. However a fairly high price must paid for that: one must admit that mathematical reasoning contains a somewhat irrationalistic (in Turing's view, unconscious) component, i.e. intuition.

This view is quite common in mathematical logic. For example Gödel maintains that, because of his incompleteness results, the "program to replace mathematical intuition by rules for the use of symbols fails" [23, p. 346]. For the solvability of all mathematical problems we need an "intuitive grasping of ever newer axioms that are logically independent from the earlier ones" [25, p. 385]. This is essential to solve problems in set theory where we have such an intuitive grasping because, "despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory". We need not "have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception". Mathematical intuition "is sufficiently clear to produce the axioms of set theory and an open series of extensions of them" [22, p. 268]. Moreover, "continual appeals to mathematical intuition are necessary not only for obtaining unambiguous answers to the questions of set theory, but also for the solution of the problems of finitary number theory (of the type of Goldbach's conjecture)" [22, p. 269]. However, despite certain similarities, sense perception and mathematical intuition are essentially different because "in the first case a relationship between a concept and a particular object is perceived, while in the second case it is a relationship between concepts" [24, p. 359]. Mathematical intuition is a form of intellectual intuition and is obtained through a process which consists in "focusing more sharply on the concepts concerned by directing our attention in a certain way". This will "produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thoughts, or grasp other basic concepts hitherto unknown to us" [25, p. 383].

The necessity of intuition and the impossibility of reducing it to logic is maintained by several other logicians. For example, Feferman claims that "the creative and intuitive aspects of mathematical work evade logical encapsulation" [12, p. 20]. Girard claims that, after Gödel's result, it is difficult to resist temptation to consider mathematical research as "essentially the research of the next axiom, in theories naturally incomplete" [20, p. 168]. And it is difficult to withstand the spell of the idea of "the *transcendency* of this next axiom" [20, p. 169]. We want to expand an incomplete theory "but there is no foreseeable way of doing it" [20, p. 168]. We usually state a new axiom after a certain experimentation but there is no theoretically justified way of doing it because "the hypothesis of a definition of the 'next axiom', whatever that means, contradicts incompleteness" [20, p. 169]. So, "there is no hope, there is, as it were, a leap in the dark, a bet at any new axiom. We are no longer in the domain of science but in that of poetry; no attempts to rationalize the world can exhaust all the more or less poetical, more or less delirious readings of Gödel" [20, pp. 169-70].

This somewhat irrationalistic view of mathematical reasoning seems to me very unsatisfactory. Can't we do better? In my opinion, in order to develop a more adequate approach, we need an alternative interpretation of the meaning of Gödel's first incompleteness theorem. The following points seem essential in such an interpretation.

1) *The development of mathematics cannot consist in proving propositions in some given formal system.* In the past century a climate of opinion was generated in which it was tacitly assumed that each area of mathematics could be supplied with a set of axioms sufficient for proving all true propositions in that area. Gödel's result showed that such an assumption is untenable because, "even if we restrict ourselves to the theory of natural numbers, it is impossible to find a system of axioms and formal rules from which, for every number-theoretic proposition A, either A or $\neg A$ would always be derivable" [25, p. 381].

2) *The development of mathematics cannot consist in building up a sequence of formal systems*. Specifically, it cannot be identified with the activity of "an idealized mathematician who entertains a sequence of successive theories, and whose theory-choices are effectively determined at each stage" [40, p. 444]. The impossibility of identifying the development of mathematics with the construction of such a sequence of theories follows from the fact that for it we can prove a result which is "an exact analogue of that of Gödel's first theorem" [40, p. 427]. This applies also when we have "an idealized mathematician whose epistemic alternatives are effectively determined at each stage, but who may have a choice among these alternatives" [40, p. 446].

3) *The development of mathematics cannot be concentrated in a single formal system but must be distributed among several systems capable of interacting*. Each system, being incomplete, must appeal to other systems for additional information, so it must be capable of communicating with them. Only by having access to the data, viewpoints and strategies of other systems a given system will be able to fill the gaps that will occur whenever its resources will turn out to be inadequate to solve a given problem.

4) *The systems forming the distributed environment cannot be formal systems.* For, introducing new information into a system while trying to generate a proof amounts to changing its axioms in the course of proof. Now in a formal system axioms cannot be changed in the course of proof because this would destroy the well-formedness of the partial proof built up at that stage. Evidence for this is provided by Prolog where using the predicates 'assert' and 'retract', which dynamically modify the data base, can completely destroy the well-formedness of the derivation. Proofs in formal systems are timeless: they are supposed to be valid for all times and places. Moreover they are acontextual: their validity can be verified simply by considering the text of the proof, independently of the context in which it occurs. Thus the terms 'now' and 'here' do not apply to them. But if, as a result of interactions with the environment, axioms can be changed in the course of proof, then proofs will no longer be timeless and acontextual and hence will no longer be formal proofs.

5) *The interactions between the systems forming the distributed environment cannot be deterministic*. In particular, they cannot be viewed (as in [3]) as applications of a composition operator that dynamically combines the axioms of several formal systems to generate a new formal system. If the interactions were deterministic, then all the systems occurring in the environment would reduce to a single system that would be subject to Gödel's result. It is often maintained that the process of proving a mathematical theorem is "nondeterministic, but nevertheless algorithmic" [56, p. 205]. In such a process the 'instructions' "are the rules of inference. The nondeterministic element is: 'to what part of the so far generated proof should I apply a new 'instruction', and what type of instruction should I apply?" [56, p. 206]. Now, this kind of nondeterminism is not sufficient to overcome the limitations revealed by Gödel's first incompleteness theorem. For, the possibility of alternative choices in applying the inference rules of an incomplete system does not make it less incomplete. The nondeterminism required by Gödel's first incompleteness theorem is the one resulting from the possibility of introducing new axioms in a nonalgorithmic way in the course of proof. This means that the process of proving a mathematical theorem must be not only nondeterministic but also nonalgorithmic.

3. Closed and open systems

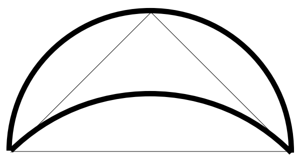
If the systems forming the distributed environment cannot be formal systems, what kind of systems can they be? In my opinion, the only answer compatible with the above interpretation of Gödel's result seems to be that they must be open systems. The distinction between open and closed systems (see [5]-[8]) can be stated as follows.

1) *Closed systems*. A closed system is a system which cannot exchange information with the environment and is based on the axiomatic method - a method by which proving a mathematical proposition means deriving it from given axioms by logical inference only. Therefore all the provable propositions of the system are contained in the axioms "as plants are contained in their seeds [14, p. 101]. Generally, "the whole of mathematics is contained in these primitive truths as in a seed" [15, pp. 204-5]. So "our only concern is to generate the whole of mathematics from this seed" [15, p. 205]. The axioms of the system cannot change in the course of proof. Moreover proof search is a finite process.

Formal systems are the most typical example of closed systems, so understandably the notion of closed system plays a crucial role in mathematical logic. Such a role is made more evident by a basic assumption of mathematical logic that may be called the *closed world view*, according to which all mathematical knowledge can be represented by closed systems. As we have seen Hilbert justified such an assumption on the ground that our understanding proceeds according to certain well-defined rules which form a closed system that can be discovered and definitively stated. Forming a closed system, they can be completely represented by formal systems. From this viewpoint assuming that the rules are complete seems quite natural. A formal system, being closed, cannot use information from other systems to prove any result, so the information implicitly contained in its axioms must be adequate to prove every true proposition in that area.

2) *Open systems*. An open system is a system which can exchange information with the environment and is based on the analytic method - a method according to which solving a problem means reducing it to another problem. The latter is temporarily assumed as hypothesis but in the end it will have to be solved similarly, i.e., by reducing it to a further problem temporarily assumed as hypothesis, and so on. Thus in the analytic method "the solution of any problem which is to be worked out is the reduction of the problem to others which are easier or already known to be within one's power" [38, p. 187]. Unlike axioms, hypotheses are not given once for all and can change in the course of proof as a result of exchanges of information with other systems. The search for a solution is a potentially infinite process because every hypothesis forms a problem that must be solved. Solving it will require a new hypothesis, and so on.

The prototype of the analytic method is given by Hippocrates of Chios' solution to the problem of the quadrature of certain lunes, e.g. the lune with an outer circumference equal to a semi-circle. Starting from a right isosceles triangle, Hippocrates draws on the hypotenuse a circular segment similar to those which appear on the legs of the triangle. In this way he obtains a lune.



Then he asks: does the lune equal the triangle? To solve this problem he introduces the hypothesis: Similar segments are as the squares on the chords. The latter entails that the larger segment equals the sums of the two smaller segments. Now, if to the mixtlinear figure inside we add the two small segments, we obtain the lune. If instead to it we add the large segment, we obtain the triangle. Therefore the lune and the triangle are equal to each other. The hypothesis that similar segments are as the squares on the chords has not been justified and must be justified. This will be done similarly, i.e., introducing a new hypothesis, and so on.

In terms of the notion of open system we may formulate an *open world view* according to which all mathematical reasoning can be represented by open systems. Each open system provides a partial representation of a certain area of mathematical knowledge and in order to get the missing information must appeal to other systems. Importing additional information from other systems does not simply result in a cumulative addition of data but may cause a radical restructuring of the system, producing changes that are global, not merely local.

Since open systems are based on the analytic method, the open world view drops the basic assumption of mathematical logic that deduction provides an adequate account of mathematical reasoning. In the analytic method deduction is only a tool for checking whether a new hypothesis is adequate. The main feature of the method is that it consists of an upward non-deductive procedure.

It seems fair to say that Gödel's first incompleteness theorem refutes the closed world view. Such a conclusion, however, is not generally accepted. For example, Gödel maintains that "everything mathematical is formalizable [although] it is nonetheless impossible to formalize all of mathematics in a *single* formal system" [21, p. 389]. Curry maintains that, although no single formal system can exhaust mathematics, the essence of mathematics lies in the formal method as such, so "mathematics is the science of formal systems" [11, p. 56]. Kleene maintains that Gödel's result does not mean that "we must give up our emphasis on formal systems. The reasons which make a formal system the only accurate way of saying explicitly what assumptions go into proofs are still cogent" [35, p. 253]. Feferman maintains that, although mathematical thought as it is actually produced is non-mechanical, this does not mean that it "cannot even be 're-represented' in mechanical terms as a result of the Gödel theorem" [13, 4.3].

The idea that everything mathematical is formalizable and that the essence of mathematics lies in the formal method as such seems to me, however, untenable because, as we have seen, by an exact analogue of Gödel's first incompleteness theorem mathematics cannot be identified with the activity of an idealized mathematician who entertains a sequence of successive theories and whose theory-choices are effectively determined at each stage. Therefore, unless I am badly mistaken, there seems to be no alternative but conclude that Gödel's first incompleteness theorem refutes the closed world view and is compatible with the open world view only.

4. The case for the analytic method

Several arguments have been used against the open world view and the analytic method. Here are some of them.

1) *The impossibility argument*. According to Hintikka and Remes the analytic method is not a method of discovery because no such method can possibly exist. A "generalized analytical method will not be an effective discovery producer". For, "in a proof of a conclusion from certain premisses we often have to consider more individuals in their relation to each other than either in the conclusion or in the premisses". Now, "in sufficiently elementary parts of geometry, their number is predictable". But this is not the case in general because "the number of additional individuals needed is often recursively unpredictable". That shows "the theoretical limitations of the 'analytical' methodology of the first great modern scientists. However powerful tool it may have been heuristically, it was not and could not have been a foolproof discovery procedure in all circumstances" [32, p. 112]. This argument is intended to show that, since there is no mechanical method for predicting the number of the additional individuals needed, the analytic method cannot provide a foolproof discovery procedure in all circumstances and so is not a proper discovery procedure. Now there is a tacit assumption in this argument, i.e. that a proper method should be infallible. But, if one supports such an assumption, then one must be prepared to admit that not only no proper method of discovery is possible but no proper method of justification is possible. For, there are several formal proofs that are too complex for their correctness to be feasibly checked.

2) *The non-algorithmicity argument*. According to Leibniz, since the analytic method provides no algorithm to find hypotheses, it cannot be considered as a proper method. For, a proper method must provide a mechanical procedure by which one "will be freed from having to think directly of things themselves, and yet everything will turn out successfully" [37, p. 256]. This argument depends on the assumption that 'rational' means 'algorithmic' and so a logic of discovery must be algorithmic. Now nobody, from the fact that provability in a given formal system cannot be decided algorithmically, would infer

that the axiomatic method cannot be considered as a method of justification. Then why, from the fact that hypotheses cannot be found algorithmically, should we infer that the analytic method cannot be considered as a method of discovery?

3) *The divination argument*. According to Robinson the analytic method is based on a procedure which is not rational insofar as it requires intuition and divination. In the analysis "the activity of my mind is not demonstration but *intuition*. The analysing geometer *divines* the premiss from which the conclusion follows" [50, p. 467]. This argument assumes that there is a basic asymmetry between the analytic and the axiomatic method. While in the latter there are rules that tell us whether a certain conclusion is derivable from certain premisses, in the former there are no rules to find premisses from which a given conclusion will follow, so the premisses can be obtained by intuition or divination only. Such an argument is just a variant of the non-algorithmicity argument. There exists a symmetry between the problem of deciding whether a certain conclusion is derivable from certain premisses and the problem of finding premisses from which a certain conclusion will follow: both problems cannot be solved algorithmically. So, as mentioned above, the non-algorithmicity argument would affect not only the analytic but also the axiomatic method.

4) *The infinite regress argument*. According to Tarski in the analysis we "have the beginning of a process which can never be brought to an end, a process which, figuratively speaking, may be characterized as an infinite regress" [54, pp. 117-8]. This argument assumes that, since the capability of our understanding is finite and hence does not permit us to cover an infinite run, the analytic method would make knowledge impossible. Now, in the analytic method at each stage we reduce the given problem to some hypothesis that may be fallible. This does not exclude the possibility of knowledge, it only excludes the possibility of reaching some ultimate absolutely justified hypothesis. So what is in question here is not the possibility of knowledge but its fallibility. The axiomatic method assumes that mathematical knowledge is infallible and so justification must depend on something which is absolutely justified. On the other hand, the analytic method recognizes that mathematical knowledge can be fallible and so there is no ultimate absolutely justified hypothesis.

5) *The locality argument*. According to Leibniz the analytic method is inferior to the axiomatic method because it is local, not global, so it does not provide a universal procedure but only one which applies to specific problems. The analytic method "goes back to the principles in order to solve the given problems only" [38, p. 232]. Therefore it is useful to solve specific problems, not to build up a whole science. Only the axiomatic method "is suitable for those who want to build up the sciences" [39, p. 90]. This argument neglects that a method, in order to be useful, must be feasible. Because of the globality of the axiomatic method, axioms are intended to solve any problem, so they can be very inefficient in solving a specific problem. As it appears from experience with automated theorem proving, the globality of the axiomatic method can make the solution of a problem extremely long and even unfeasible. On the other hand, the locality of the analytic method is one of the reasons for its efficiency.

6) *The modularity argument*. Accoding to Leibniz, unlike the axiomatic method, the analytic method does not permit to reuse previously obtained results. In solving each problem it goes back to principles "just as if neither we nor others had discovered anything before" [38, p. 232]. Therefore, when we apply analysis to a particular problem, "we often do work that has already been done". On the contrary, the axiomatic method permits to reuse already done work. So "it is more important to establish syntheses, because this work is of permanent value" [38, p. 233]. This argument, which concerns the modularity of proof, seems to be an oversight. Not only in the analytic method we may reuse already done work, but in a sense the analytic method is even more modular than the axiomatic method. In the latter, in order to make a proof really modular, one would have to isolate

the role of the various axioms. This is generally impossible because in several cases the roles of the axioms within a given proof are so intertwined that they cannot be sharply distinguished. This is due to the fact that axioms are universal, so they are intended to solve every problem, and in order to be manageable must be limited in number. On the contrary, in the analytic method proofs, being built upwards, can be easily modularized.

Contrary to what the above criticisms are intended to imply, the analytic method provides an approach to mathematical reasoning more realistic than the axiomatic method. The latter does not account for the basic feature of mathematics of using “a motley of techniques of proof” [59, p. 176]. On the other hand such a feature is well accounted for by the analytic method. Unlike the axiomatic method, it does not pretend to be a universal procedure capable of solving any problem but only tries to make the most of the specific features of each problem.

5. The meaning of analysis

In terms of the distinction between closed and open systems, Gentzen’s sequent or natural deduction systems, being formal systems, are closed systems. This seems to contradict Hintikka and Remes’ claim that there is a strict relationship between the analytic method, on which open systems depend, and “certain relatively new techniques in symbolic logic which may be called *natural deduction methods*” [33, p. 253]. By ‘natural deduction methods’ Hintikka and Remes mean Gentzen’s cut-free sequent calculus and Beth’s tableaux.

According to Hintikka and Remes the analytic method “is almost a special case of these natural deduction systems” [33, p. 253]. In their view, “a proof obtained by means of the analytical method amounts essentially to a proof by the so-called natural deduction methods” [32, p. *xiii*]. In particular, “the logic of the method satisfies the so-called subformula property, which is the characteristic feature of natural deduction methods” [33, p. 253]. Even “the considerable hesitancy and uncertainty which there seems to have prevailed among ancient geometers concerning the precise nature of the justification of analysis has a solid systematical reason in the difficulty and subtlety of the permutation rules of modern proof theory” [33, p. 273]. Hintikka and Remes acknowledge that there are substantial differences between the analytic method and natural deduction methods but claim that “these discrepancies can be understood as being due to the exigencies of the task of a practicing mathematician”. In their view, “even if ancient geometers had known natural deduction methods and tried to use them strictly, these practical difficulties would have pushed them to the familiar form of the traditional methods” [33, p. 254].

Such claims are not incompatible with the fact that Gentzen’s cut-free sequent calculus or Beth’s tableaux are closed systems. Simply, they refer to a different version of the analytic method which is not alternative but subordinated to the axiomatic method. There are two different versions of the analytic method. According to one version, supported by Plato, Descartes, Newton and more recently Lakatos, the analytic method is alternative to the axiomatic method. According to another version, supported by Pappus, Galilei, Leibniz, Kant and more recently Polya and Hintikka, the analytic method is subordinated to the axiomatic method: it is only a heuristic method which serves to discover proofs within a given axiomatic system. Of course only the first version of the analytic method is compatible with the open world view. The second version, being just a subsidiary of the axiomatic method, is part of the closed world view.

These two versions of the analytic method correspond to two different senses of analysis. In the second version analysis means the passage from a compound to ingredients. This is Frege’s sense when he compares the analysis of a question to the chemical analysis of a body. Frege states that, “as it is not immaterial whether or not I carry out a chemical analysis of a body in order to see what elements it is composed of, so it is not immaterial whether I carry out a logical analysis of a logical structure in order to find out what its constituents are or leave it unanalysed as it were simple, when it is in fact complex" [15, pp. 208-9]. Clearly here analysis means resolution of a complex structure into simpler parts.

This is also Hintikka and Remes' sense of analysis when they claim that the logic of the analytic method must satisfy the subformula property - a property which must be "present if in the process of looking for a proof formulas are all the time chopped into simpler ones" [33, p. 267]. This is, moreover, Prawitz's sense of analysis when he describes a normal derivation as consisting of "two parts, one *analytical part* in which the assumptions are broken down in their components by use of the elimination rules, and one *synthetical part* in which the final components obtained in the analytical part are put together by use of the introduction rules" [47, p. 249]. Prawitz's description closely corresponds to Polya's description of the procedure through which proofs are generated by the second version of the analytic method: "You decompose the whole into its parts, and you recombine the parts into a more or less different whole" [45, p. 76]. According to Polya we first try "to *separate the various parts of the condition*, and to examine each part by itself" [45, p. 77]. We decompose the condition "gradually, but no further than we need to" [45, p. 76]. Then, "after having decomposed the problem, we try to recombine its elements in some new manner" [45, p. 77].

This sense of analysis is somewhat limited because it reduces analysis to a purely mechanical business. Proof search in Gentzen's cut-free sequent calculus or in Beth's tableaux is a mechanical process and indeed, starting from the pioneer work of Prawitz [48], this fact has been exploited in several systems of automated theorem proving. The same applies to proof search in Gentzen's natural deduction systems (see [52], [55]). Therefore, assimilating analysis to resolution of a complex structure into simpler parts reduces the search for hypotheses to a purely mechanical business.

This, however, is not the only sense of analysis. One must "distinguish between analysis into concepts and division into parts" [38, p. 665]. Analysis into concepts is the main concern of the first version of the analytic method where analysis is a non-mechanical process. To analyse a problem means to find hypotheses for its solution, and to analyse a proposition means to find conditions for its truth. One may argue that this sense of analysis is broader than the one which identifies analysis with resolution of a complex structure into simpler parts. This appears from its features.

1) *It does not assume that hypotheses should be contained in the problem as its parts.* In order to analyze a complex problem we must often relate its attributes "to other attributes as well as to the subjects which contain them". The relating of an attribute to others makes apparent "their concurrence in the same subject, their connection with each other, their compatibility, and on the other hand, how one can be changed into another or can be produced out of several others" [38, p. 287]. Since the additional attributes which must be considered need not be contained in the given problem, hypotheses involve considerations external to it.

2) *It does not assume that hypotheses should be simpler than the problem*. This applies also when analysis is understood as resolution of a complex structure into simpler parts. For, as Leibniz points out, "parts are not always simpler than wholes, though they are always less than the whole" [38, p. 665]. The example Leibniz has in mind here is presumably that of the infinitesimals which, while being part of the continuum, are not simpler than it. That hypotheses need not be simpler than the problem is even more true when analysis is understood as the process of findind hypotheses for solving a given problem. For then complex hypotheses may be necessary even to solve comparatively simple problems in elementary number theory.

3) *It does not assume that hypotheses should be less certain than the problem*. For example Zermelo justifies the axiom of choice on account of the fact that it is required to

prove certain apparently uncontroversial propositions, whence he concludes that "the principle is necessary for science" [60, p. 187]. Similarly Gödel wants to justify new large cardinal axioms on account of their success, where success means "fruitfulness in consequences, in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover" [22, p. 261].

6. Abduction

I have claimed that, in view of Gödel's first incompleteness theorem, mathematical reasoning cannot be reduced to deduction and must be analyzed in terms of open systems which are based on the analytic method. There is nothing really new in such an assumption because in the past the analytic method has been often considered as the model of the process of mathematical discovery. In particular such a view was strongly supported by the first great modern scientists (Galilei, Descartes, Newton) who claimed that they had found their results by means of the analytic method. For example Newton states: "The Propositions in the following book [*Principia*] were invented by Analysis. But considering that the Ancients (so far as I can find) admitted nothing into Geometry before it was demonstrated by Composition, I composed what I invented by Analysis to make it Geometrically authentic and fit for the publick" [42, p. 294]. So Newton considered the axiomatic presentation as just a concession to tradition and to the tastes of the public.

That does not mean that I intend to claim that the analytic method is the logic of mathematical discovery. As the axiomatic method is not the logic of justification but only an object of study for it, so the analytic method is not the logic of discovery but only an object of study for such a logic. To develop a logic of mathematical discovery one must start from the analytic method but must also supplement it with something not directly provided by that method. What the analytic method does not provide is an indication as to the processes through which hypotheses are to be found. Again, the situation is similar to that of the axiomatic method. The latter defines a certain notion of proof - the axiomatic notion of proof - but does not provide any indication as to how to find a proof for a given proposition from given axioms. Similarly the analytic method defines a certain notion of proof - the analytic notion of proof - but does not provide any indication as to how one could find hypotheses to solve a given problem.

There are several processes to find hypotheses. Here I will have time only to touch on a couple of them, abduction and analogy, omitting other important processes such as induction whose importance in this respect has been also recently reasserted (see [1], [19]).

Abduction is currently understood as characterized by the following reasoning pattern: given the sentences $A \rightarrow C$ and C, derive A as an explanation of C [10, p. 661]. This is just Peirce's [43, 5.189] original form of abduction. Such a characterization seems, however, unsatisfactory insofar as in abductively inferring the hypothesis A we must use a premiss, $A \rightarrow C$, where the hypothesis A already occurs in the antecedent. This trivializes abduction as a "process of forming an explanatory hypothesis" [43, 5.171]. The hypothesis is not formed as a result of the abductive inference but is presupposed by it. Much of the difficulty with Peirce's view of abduction stems from the fact that he considers abduction as an inference. In view of this it seems better to express it as the following problem: given a set of sentences Γ and a sentence C not derivable from Γ, to find a sentence A such that C is derivable from $\Gamma+A$. The set Γ consists of the already available hypotheses, or background hypotheses, C is the problem to be solved, and A is the new hypothesis sought for.

Clearly not every A would provide an interesting explanation of C. To begin with, $\Gamma+A$ must be consistent. Moreover, given Γ, the hypothesis A must be more economical than its envisioned competitors. This is already pointed out by Peirce who, in view of the fact that A is only one out of several possible explanations, assigns an important role to "the consideration of economy" [43, 7.220]. Thus his form of abduction should be modified as follows: given the sentences $A \rightarrow C$ and C, if A is more economical than its envisioned competitors (i.e., other sentences B such that $B \rightarrow C$), derive A as an explanation of C. In order to see that A must be more economical than its competitors, let Γ be Peano's axioms less the axiom of induction, A the conjunction of the axioms of (some finite axiomatization of) set theory, and C the sentence $\forall x \forall y(x+y=y+x)$. Then, given the background hypotheses Γ, the hypothesis A would certainly provide an explanation for C but a not very illuminating one because it would not show which specifically airthmetic property is involved in C. A more interesting solution is obtained by letting A be the axiom of induction.

This suggests that the abduction problem should be stated in terms of a preference relation $A \subseteq B$ (A is preferred to B). For any set of sentences Γ and any sentence C not derivable from Γ, let $\Theta(\Gamma,C)$ be the set of all sentences B such that $\Gamma+B$ is consistent and C is derivable from $\Gamma+B$. Then the abduction problem can be expressed as follows: given a set of sentences Γ and a sentence C not derivable from Γ, to find a sentence A in $\Theta(\Gamma,C)$ such that $A \subseteq B$, for any other sentence B in $\Theta(\Gamma,C)$. Usually the preference relation is expressed in terms of logical consequence: $A \subseteq B$ whenever B is a logical consequence of A. It is usually claimed that such a preference criterion, albeit very weak, "has the adavantage that its definition is logically supported and homogeneous with the object-level definition of abduction" [10, p. 673]. W.r.t. such preference criterion the abduction problem is solvable in a number of cases (see [4], [9], [49], [53]) where essentially abduction reduces to deduction. The fact remains, however, that the logical consequence criterion is a very weak one. The abduction problem requires criteria that would be termed non-logical by mathematical logicians. So the whole subject requires further investigation.

7. Analogy

Current approaches to abduction, while useful for the second version of the analytic method, are of limited use for the first version because they ultimately reduce abduction to deduction. Other procedures seem to be necessary to discover hypotheses. In that respect an especially important role is likely to be played by analogical reasoning, i.e. reasoning from known similarities between two things to the existence of further similarities. The importance of analogical reasoning for the analytic method is already stressed by Leibniz according to whom the analytic method "rests for the most part upon analogies" [38, p. 284].

The central role of analogical reasoning in the analytic method is illustrated by the Hippocrates of Chios' solution to the problem of the quadrature of the lune with an outer circumference equal to a semi-circle. Hippocrates states the crucial hypothesis that similar segments are as the squares on the chords, on the analogy of the fact that circles are to one another as the squares on the diameters. He argues that, "as the circles are to one another, so are the similar segments. For similar segments are those which are the same part of the circle, e.g., semicircle is similar to semicircle and the third part of a circle to the third part. Therefore also, similar segments admit equal angles. For the angles of all semicircles are right, and those of greater segments are less than right and as much less as the segments are greater than semicircles; and the angles of segments less than a semicircle are greater and as much greater as the segments are less" [Simplicius, *In Aristotelis Physica comment.*, 61.5-18].

While analogy often plays a crucial role in finding a hypothesis, it does not justify it. Afterwards, however, the hypothesis can be made respectable by showing that one can actually derive the desired solution from it. For example, in the case of Hippocrates, from the fact that circles are to one another as the squares on the diameters, one can derive that similar segments are as the squares on the chords. The analogy is essential to state the hypothesis, which however is tested and corroborated by a purely deductive argument.

There are at least two different forms of analogical reasoning because there are at least two different senses of similarity. According to one sense, two things are similar if they have a certain likeness. For this sense of similarity we may state the following rule: If $P(a)$, and b is like a, then $P(b)$. This also includes Aristotle's sense of similarity according to which similarity occurs "when, given four terms, the second, *B*, stands with the first one, *A*, in the same ratio as the fourth, *D*, stands with the third one, *C*" [Aristotle, *Poetics*, 1457 b, 16-17]. This is usually viewed as the similarity of two relations (A:B and C:D) but can be also interpreted as follows: as *B* has a certain likeness to *D*, *A* has a certain likeness to *C*. This sense of similarity is the one used by Hippocrates. He defines similar segments as those which are the *n*-the part of their circles, for some *n*. This includes the case n=1, when the segment consists of the whole circle. Segments have a certain likeness to circles, and the chords of circles have a certain likeness to the diameters of circles. This suggests that, since circles are to one another as the squares on the diameters, similar segments should be to one another as the squares on the chords.

According to another sense of similarity, two things are similar if they partially agree on certain attributes. For this sense of similarity we may state the following rule: If $P(a)$, and b and a agree on attributes $Q_1(x),\ldots,Q_k(x)$, then $P(b)$. This rule is implicit in Kant who expresses it as follows: "*Analogy* concludes from *partial* similarity of two things to *total* similarity according to the principle of *specification*: Things of one genus which we know to agree in much, also agree in the remainder as we know it in some of the genus but do not perceive it in others" [34, pp. 136-137]. In addition to the attributes on which two things agree one may also consider those on which they disagree. Then the previous rule can be extended as follows: If $P(a)$, and b and a agree on attributes $Q_1(x),\ldots,Q_k(x)$ but disagree on attributes $Q_{k+1}(x),\ldots,Q_{k+m}(x)$, then $P(b)$.

All such similarity rules are non-deductive in that the conclusion does not soundly follow from the premisses and depends on information not provided by the premisses. So the problem arises of specifying knowledge that, when added to the premisses, will make the conclusion follow soundly. Such additional knowledge, however, will consist of implicit or *tacit* knowledge: it will remains idle most of the time and will be used only at certain crucial times. The fact that such additional knowledge will remain tacit most of the time explains the efficiency of the analogical reasoning in finding hypotheses. Making all such knowledge fully explicit would hinder the search for hypotheses.

Incidentally, tacit knowledge is essential not only to find new hypotheses and hence new proofs, but also to understand proofs already built up by others. When we read a proof we make a number of tacit assumptions concerning the use, notation and interpretation of concepts occurring in it, concerning the way arguments are expressed and also the use of diagrams as a means to support the arguments. To make such assumptions fully explicit would make the proof extremely complex and ultimately incomprehensible. Tacit knowledge allows to understand and communicate proofs in a feasible way.

In analogical reasoning tacit knowledge is handled by means of some sort of selective ignorance. Developing the machinery for selecting the relevant knowledge when it is needed while ignoring knowledge which is not needed, is a subject for future research.

8. Concluding remarks

Boole's idea that logic is the study of the laws of thought has been implemented by mathematical logic by identifying mathematical reasoning with deduction. Peirce's idea that the operations of reason must be analyzed reducing reasoning to its ultimate elements has been implemented by Gentzen and Prawitz through the calculus of natural deduction. However, identifying mathematical reasoning with deduction is impossible in view of Gödel's first incompleteness theorem which seems only compatible with the open world view. This means that deduction must be considered as just a component of mathematical reasoning - a component whose main function is to test the validity of hypotheses. Several other components must be taken into account, such as abduction, analogy, induction, the role of tacit knowledge and so on. All such components must be thoroughly investigated in order to provide an adequate account of mathematical reasoning.

In this century a sharp distinction has been made between the justification of mathematical knowledge and mathematical discovery. This depends on the assumption that the latter merely consists of the initial conception of a new idea. On the contrary, mathematical discovery includes the long process through which the initial idea is tested and revised, the revised idea is tested and revised, and so on until a possible solution is found. Thus, when properly understood, the justification of mathematical knowledge is only a part of the process of mathematical discovery. All the methods of mathematics are directed toward the simultaneous invention and justification of ideas. Therefore, if Boole's idea that logic is the study of the laws of thought is to be taken seriously, analyzing both invention and justification must be the main concern of logic.

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