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Critical comments are welcome

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Remaking Logic

What Is Logic, *Really*?

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Preface

Despite strenuous efforts by its proponents, mathematical logic has generally failed to convince mathematicians, natural scientists and human scientists of its relevance to their work, increasingly so in the last few decades. Why does this come about and how to remedy it? This lack of regard for mathematical logic contrasts with the consideration logic enjoyed in antiquity, not only as one of the three main parts of philosophy, but especially as supplying the sciences with its instruments. Indeed, almost no one disputed the idea that such instruments could be beneficial to their potential users, so much so that logic was made a standard part of the school curriculum.

The aim of this book is to deal with these questions. To this purpose, the book first gives an overview of how logic and its relation with the scientific method have been conceived in antiquity and in the modern age, since this provides useful indications for a new approach to the subject. Then it proposes a new view of logic and its relation with evolution, language, reason, method and knowledge, as well as a new view of philosophy and its relation with knowledge, since seeing logic in a wider perspective helps to place the subject on a more satisfactory basis.

Contents

Introduction

1. *The Intended Purpose of Mathematical Logic*

In 1931 the question ‘What is logic, really?’ would have received a straightforward answer: Logic is mathematical logic. For by that time mathematical logic had completely superseded traditional Scholastic logic.

In a book published in that year, Scholz expressed his enthusiasm for the new form of logic invented by Frege, claiming that it “gives us the complete inferential rules which the development of the tremendously exacting modern mathematics requires.”[1]

Scholz’s claim is so much more significant as it was made the very same year that Gödel published his incompleteness theorems, which were ruinous for the intended aim of mathematical logic: To give a secure foundation for mathematics.

2. *The Basic Assumptions of Mathematical Logic*

That this was the intended aim of mathematical logic is apparent from Frege, who bases the subject on the following assumptions.

1) *The aim of mathematical logic is to give a secure foundation for mathematics*. Mathematical logic pursues this aim through the study of the method of mathematics, which will show “the ultimate ground upon which rests the justification for holding” a proposition “to be true.”[2]

2) *The method of mathematics is the axiomatic method*. Almost “the entire activity of the mathematician consists in drawing” deductive “inferences.”[3] We start from axioms “expressly declared as such, so that we can see distinctly what the whole structure rests upon,” and proceed from them by rules of deduction “specified in advance.”[4]

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[1] Scholz 1961, 67.
[2] Frege 1959, 3.
[3] Frege 1979, 203.
[4] Frege 1964, 2.

3) *Mathematical logic is the study of the axiomatic method, starting from deduction*. For studying the “laws of valid inference is the goal of logic.”[5]

4) *Mathematical logic need only express what is necessary for the axiomatic method*. It may “forgo expressing anything that is without significance for the inferential sequence.”[6] Namely, “anything that it is not necessary for setting up the laws of deduction.”[7] Everything “necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated.”[8] Thus mathematical logic will fail to express all aspects of mathematics. But mathematical logic “is a device invented for certain scientific purposes,” namely, to give a secure foundation for mathematics, “and one must not condemn it because it is not suited to others.”[9]

5) *Mathematical logic can actually give a secure foundation for mathematics*. Indeed, mathematics is “merely a more highly developed logic.”[10] Admittedly, to the question why the primitive laws of logic, and hence of mathematics, are true, mathematical logic “can give no answer.”[11] Trying to answer this question by means of it “would be as much as to judge without judging, or to wash the fur without wetting it.”[12] But intuition guarantees that the primitive laws of logic, and hence of mathematics, are true.[13]

Assumptions 1)–5) are the basic assumptions of mathematical logic, by which Frege shaped the discipline. Clearly, the subject, as presented in current textbooks, is not a direct descendant of Frege but rather of Hilbert and his school.[14] Nevertheless, it is founded on Frege’s basic assumptions, as it is apparent from its leading representatives, such as Gödel and Tarski, who state:

1) The aim of mathematical logic is to give a secure “foundation for mathematics.”[15] Mathematical logic pursues this aim through the study of the method of mathematics, which will assure for mathematics the “highest possible degree of clarity and certainty.”[16]

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[5] Frege 1979, 3.
[6] Frege 1967, 6.
[7] Frege 1979, 5.
[8] Frege 1967, 12.
[9] *Ibid*., 6.
[10] Frege 1964, 3.
[11] *Ibid*., 15.
[12] Frege 1959, 36.
[13] On the role of intuition in Frege, see Chapter 9. Unless otherwise stated, here and in what follows ‘intuition’ means ‘intellectual intuition’, that is, the immediate apprehension of intellectual objects, not to be confused with Kant’s pure sensible intuition, which concerns spatio-temporal appearances.
[14] Especially Hilbert and Ackermann 1950 and Hilbert and Bernays 1968–1970.
[15] Gödel 1986–2002, III, 45.
[16] Tarski 1994, 109.

2) The method of mathematics is the axiomatic method, because every mathematical "discipline begins with a list of a small number of sentences, called axioms" which "are recognized as true without any further justification."[17] Then "no other sentence is accepted in the discipline as true unless we are able to prove it with the exclusive help of axioms and those sentences that were previously proved."[18]

3) Mathematical logic is the study of the axiomatic method, starting from deduction, because "every mathematical discipline" is "a deductive theory."[19] And mathematical logic is "the methodology of deductive sciences."[20]

4) Mathematical logic need only express what is necessary for the axiomatic method, because it serves "exclusively for the purpose of studying deductive sciences."[21] Thus mathematical logic will fail to express all aspects of mathematics. In particular, the idea "to replace mathematical intuition by rules for the use of symbols fails."[22]

5) Mathematical logic can actually give a secure foundation for mathematics, because "the problem of giving" a secure "foundation for mathematics" falls "into two different parts."[23] First, the "methods of proof have to be reduced to a minimum number of axioms" and deductive "rules of inference."[24] This is obtained through "the so-called 'formalization' of mathematics."[25] Secondly, "a justification in some sense or other has to be sought for these axioms."[26] Admittedly, such a justification cannot be given by mathematical logic. But it can given by intuition, since axioms "can directly be perceived to be true (owing to the meaning of the terms or by an intuition of the objects falling under them)."[27]

3. *Inadequacy of the Basic Assumptions of Mathematical Logic*

The basic assumptions of mathematical logic, however, are inadequate, as it is apparent from Gödel's incompleteness theorems.[28]

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[17] Tarski 1969, 70.
[18] *Ibid.*
[19] Tarski 1994, 112.
[20] *Ibid.*, 130.
[21] Tarski 1983, 166.
[22] Gödel 1986–2002, III, 346.
[23] *Ibid.*, III, 45.
[24] *Ibid.*
[25] *Ibid.*
[26] *Ibid.*
[27] *Ibid.*, III, 347.
[28] For a precise statement of Gödel's incompleteness theorems see, for example, Cellucci 2007, 175–178, 184–185 (Gödel's first incompleteness theorem), 180–184 (Gödel's second incompleteness theorem), 185–187 (Gödel's third incompleteness theorem).

1) The assumption that the aim of mathematical logic is to give a secure foundation for mathematics conflicts with the facts 2)–5) below, which show that this aim cannot be achieved.

2) The assumption that the method of mathematics is the axiomatic method is refuted by Gödel's first incompleteness theorem. By the latter, for any consistent, sufficiently strong, deductive theory *T*, there is a sentence *G* of *T* which is true, specifically, true in the natural numbers, but indemonstrable in *T*.[29] Thus mathematical reasoning cannot be reduced to deductive reasoning and the method of mathematics cannot be identified with the axiomatic method. It is no way out to identify mathematics with an infinite sequence of axiomatic systems, rather than a single one, on the ground that, by Gödel's first incompleteness theorem "the concept of intuitively valid proof cannot be exhausted by any single formalization."[30] For then the method of mathematics would be the axiomatic method plus something which transcends it, because a mathematical demonstration would be "that sort of growing thing which the intuitionists have postulated for certain infinite sets."[31] But, that a demonstration is a growing thing is incompatible with the axiomatic method, where a demonstration is a fixed thing. It is compatible only with the analytic method; see Chaper 3.

3) The assumption that mathematical logic is the study of the axiomatic method, and hence the study of the method of mathematics, conflicts with 2) above, by which the axiomatic method cannot be said to be the study of the method of mathematics.

4) The assumption that mathematical logic need only express what is necessary for the axiomatic method, once again conflicts with 2) above, by which, expressing what is necessary for the axiomatic method, mathematical logic will fail to express what is necessary for the method of mathematics.

5) The assumption that mathematical logic can actually give a secure foundation for mathematics is refuted, first, by the fact that, by 2) above, the methods of demonstration cannot be reduced to a minimum number of axioms and deductive rules of inference; secondly, by the fact that intuition, being subjective and unreliable, cannot guarantee that the axioms are true. Nor there can be any other absolute guarantee that the axioms are true. By Gödel's second incompleteness theorem, for any consistent, sufficiently strong, deductive theory *T*, the sentence canonically expressing the consistency of *T*, Con(*T*), is indemonstrable in *T*, hence *a fortiori* indemonstrable by absolutely reliable means.[32]

Actually, the view that there could be a secure foundation for mathematics was no more reasonable than the Hindu's view, mentioned by

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[29] Throughout this book, 'deductive theory' means 'axiomatic formal theory', and 'sufficiently strong deductive theory' means 'deductive theory containing first-order Peano arithmetic'.

[30] Curry 1977, 15.

[31] *Ibid.*, 15.

[32] A sentence canonically expressing the consistency of a deductive theory is one satisfying the conditions of Löb 1955. A simpler condition, however, is sufficient; see Cellucci 2007, 180–181.

Russell, “that the world rested upon an elephant and the elephant rested upon a tortoise; and when they said, ‘How about the tortoise?’ the Indian said, ‘Suppose we change the subject’.”[33]

4. *The Reception of the Limitative Results*

That Gödel’s incompleteness theorems show that the basic assumptions of mathematical logic are inadequate is something several mathematical logicians, and even mathematicians, are unwilling to accept.

For example, Murawski states that, although Gödel’s incompleteness theorems indicate “certain weaknesses of the axiomatic-deductive method,” the latter makes it possible to specify the notion of demonstration and “the correctness of methods admissible in mathematics.”[34] If we resigned from it, the notion of demonstration “would become entirely subjective.”[35]

This is unwarranted. It is unjustified to say that the axiomatic-deductive method makes it possible to specify the notion of demonstration. By Gödel’s first incompleteness theorem, the axiomatic notion of demonstration is inadequate for mathematics. It is unjustified to say that the axiomatic-deductive method makes it possible to specify the correctness of methods admissible in mathematics. By Gödel’s second incompleteness theorem, there is no absolutely reliable way of establishing their correctness. It is unjustified to say that, if we resigned from the axiomatic-deductive method, the notion of demonstration would become entirely subjective. From antiquity, there has been an alternative notion of demonstration, the analytic one, which is by no means subjective and, in addition, is not affected by Gödel’s incompleteness results; see Chapter 16.

As another example, Gowers states that, as a result of “one of the great discoveries of the early 20th century, largely due to Frege, Russell, and Whitehead,” every mathematical demonstration can be expressed as a deduction which “starts with axioms that are universally accepted and proceeds to the desired conclusion by means of only the most elementary logical rules.”[36] This “means that any dispute about the validity of a mathematical” demonstration “can always be resolved.”[37] Admittedly, there remains the question “why should one accept the axioms proposed by mathematicians,” but “what a mathematical demonstration actually does is show that certain conclusions” follow “from certain premises.”[38] The “validity of these premises is an entirely independent matter which can safely be left to philosophers.”[39]

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[33] Russell 1996, 4.

[34] Murawski 1999, 338.

[35] *Ibid.*

[36] Gowers 2002, 39–40.

[37] *Ibid.*, 40.

[38] *Ibid.*, 41.

[39] *Ibid.*

This is unwarranted. It is unjustified to say that every mathematical demonstration can be expressed as a deduction which starts with axioms that are universally accepted and proceeds to the desired conclusion by means of only the most elementary logical rules. As we will see in Chapter 11, by the strong incompleteness theorem for second-order logic, this is impossible. It is unjustified to say that any dispute about the validity of a mathematical demonstration can always be resolved. By Gödel's second incompleteness theorem, there is no absolutely reliable way of establishing its validity, so mathematics cannot be said to be absolutely certain. It is unjustified to say that what a mathematical demonstration actually does is show that certain conclusions follow from certain premises. By Gödel's first incompleteness theorem, mathematical demonstration cannot be reduced to deduction. It is unjustified to say that the validity of the premises is a matter which can safely be left to philosophers – meaning that it is irrelevant to mathematics. For if the premises were inconsistent, any conclusion would follow. Therefore, if mathematics is not to be a vacuous exercise, the validity of the premises is essential.[40]

A notable exception to this attitude toward Gödel's incompleteness theorems is Post. Already in 1941 he expressed his "continuing amazement that, ten years after Gödel's remarkable achievement," the "current views on the nature of mathematics are thereby affected only to the point of seeing the need of many formal systems, instead of a universal one."[41] On the contrary, "has it seemed to us to be inevitable" that Gödel's achievement "will result in a reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries."[42] Axiomatic "thinking will then remain as but one phase of mathematical thinking."[43]

5. *Mathematics and Axiomatic Formal Theories*

It has been claimed above that the method of mathematics cannot be identified with the axiomatic method because, by Gödel's first incompleteness theorem, for any consistent, sufficiently strong, deductive theory T, there is a sentence G of T which is true, specifically, true in the natural numbers, but indemonstrable in T.

Against this claim the following objection could be raised. The above statement of Gödel's first incompleteness theorem involves the notion of truth in the natural numbers. This amounts to assuming that natural numbers are accessible to knowledge and that the notion of truth in the natural numbers is clear, but this assumption is metaphysical and unjustified. What Gödel actually shows, as later improved by Rosser, is the following 'syntactic' result: For any consistent, sufficiently strong, deductive theory T, there is a sentence G of T, specifically, an arithmetical sentence, such that both G and $\neg G$ are indemonstrable in T.

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[40] This is apparent from the answer to objection 3) in the next section.

[41] Post 1965, 345.

[42] *Ibid.*

[43] *Ibid.*

For example, Floyd and Putnam state: "That the Gödel theorem shows that (1) there is a well-defined notion of 'mathematical truth' applicable to every formula of" the system of *Principia Mathematica* "*PM*; and (2) that, if *PM* is consistent, then some 'mathematical truths' in that sense are undecidable in *PM*, is not a mathematical result but a metaphysical claim."[44] Conversely, the above mentioned syntactic result "is precisely the mathematical claim that Gödel proved."[45]

But this objection is unjustified for at least two reasons.

First, for any consistent, sufficiently strong, deductive theory *T*, the sentence *G* and the sentence canonically expressing the consistency of *T*, Con(*T*), are equivalent, and their equivalence can be demonstrated in *T*. Therefore we have as much, or as little, reason to believe that *G* is true as we have reason to believe that *T* is consistent. Thus, if we know that the deductive theory *T* is consistent, we also know that the sentence *G* is true. Then the truth of *G* is already implicit in the assumption that *T* is consistent.[46]

Secondly, a syntactic deductive theory *T* capable of talking about demonstrations in *T* – as required by the construction of the sentence *G* – will need to talk about expressions. Now, the syntax of expressions and arithmetic are interpretable into each other.[47] Therefore, anyone who had reservations about the notion of natural number ought to have reservations about syntax.

The objection depends on the assumption that, if natural numbers are accessible to knowledge, thereby all their properties will be so accessible. This assumption is as unwarranted as the assumption that, if a painter makes a painting, thereby he will know all properties of his painting. The latter would imply that the art critics are useless, which is absurd even if some painter believe it.[48]

6. *Mathematics and the Loss of Certainty*

It has been claimed above that mathematics cannot be said to be absolutely certain because, by Gödel's second incompleteness theorem, for any consistent, sufficiently strong, deductive theory *T*, the sentence canonically expressing the consistency of *T*, Con(*T*), is indemonstrable in *T*, hence *a fortiori* indemonstrable by absolutely reliable means.

Against this claim the following objections could be raised.

1) If mathematics cannot be said to be absolutely certain, then Gödel's second incompleteness theorem, being a mathematical result, cannot be said to be absolutely certain. But the claim that mathematics cannot be said to be absolutely certain is based on Gödel's result. Thus this claim cannot be said

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[44] Floyd and Putnam 2000, 632.

[45] *Ibid.*

[46] This assumption is essential because, if *T* is inconsistent, *G* is false, and demonstrable in *T* since then every sentence is.

[47] See, for example, Quine 1946.

[48] For example, Boccioni states that "the art critics are useless or harmful" (Boccioni 1997, 208).

to be absolutely certain. Therefore, the claim that, by Gödel's second incompleteness theorem, mathematics cannot be said to be absolutely certain, is self-defeating.[49]

But this objection is unjustified, because the argument that, by Gödel's second incompleteness theorem, mathematics cannot be said to be absolutely certain, does not depend on the assumption that Gödel's second incompleteness theorem can be said to be absolutely certain. It is a reduction to the impossible since it is of the following kind. Let us suppose, for argument's sake, that mathematics can be said to be absolutely certain. Then Gödel's second incompleteness theorem, being a mathematical result, can be said to be absolutely certain. But, by Gödel's second incompleteness theorem, mathematics cannot be said to be absolutely certain. So mathematics cannot be said to be absolutely certain. Contradiction. Therefore mathematics cannot be said to be absolutely certain.

2) Gödel's second incompleteness theorem is irrelevant to the question of the absolute certainty of mathematics. For, either (A) "we have no doubts about the consistency of" Zermelo-Fraenkel set theory with the axiom of choice ZFC, in which case "there is nothing in the second incompleteness theorem to give rise to any such doubts."[50] Or (B) "we do have doubts about the consistency of ZFC," but then "we have no reason to believe" that a demonstration of the consistency of ZFC "formalizable in ZFC would do anything to remove those doubts."[51] For "the consistency of ZFC is precisely what is in question."[52]

But this objection is unjustified, because case (A), that we have no doubts about the consistency of ZFC, might occur only if we could give a demonstration of the consistency of ZFC by absolutely reliable means. But, by Gödel's second incompleteness theorem, this is impossible. On the other hand, case (B) is misstated. For if we do have doubts about the consistency of ZFC, then the question is not whether a demonstration of the consistency of ZFC formalizable in ZFC would do anything to remove those doubts. It is, rather, whether a demonstration of the consistency of ZFC given by absolutely reliable means would do anything to remove them. And the answer is yes, definitely.

That, by Gödel's second incompleteness theorem, a demonstration of the consistency of ZFC given by absolutely reliable means is impossible, implies that the doubts about the consistency of ZFC cannot be removed. Then it is unjustified to say that "nothing in Gödel's theorem is in any way incompatible with the claim that we have absolutely certain knowledge of the truth of the axioms" of "any of the formal systems that we use in mathematics," and "therewith of their consistency."[53] By Gödel's second incompleteness theorem, we are not entitled to make this claim.

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[49] This objection was raised in correspondence by Reuben Hersh, acting as *advocatus diaboli*, not because he shared it.

[50] Franzén 2005, 105.

[51] *Ibid*., 105–106.

[52] *Ibid*., 105.

[53] *Ibid*.

3) The impossibility of demonstrating the consistency of the axioms by any absolutely reliable means, established by Gödel's second incompleteness theorem, is not incompatible with the claim that mathematics is absolutely certain, for we need not accept the rule that one can draw any conclusion from a contradiction. Rather, we may adopt Wittgenstein's rule: "Don't draw any conclusions from a contradiction: make that a rule."[54] Then, if we get to a contradiction, we will "simply say, 'This is no use – and we won't draw any conclusion from it."[55] Thus the contradiction is sealed off, and a "contradiction is harmless if it can be sealed off."[56]

But this objection is unjustified. Wittgenstein's rule is no solution, because it does not remove the cause of a contradiction. It merely makes the contradiction artificially harmless by avoiding to draw any conclusion from it. Contradictions usually arise from use of poorly formulated concepts, and their discovery is a powerful incentive to reformulate them. For example, the paradoxes of the calculus led to reformulate the concept of infinitesimal, and Russell's paradox the concept of set. On the contrary, Wittgenstein's rule leaves everything as it is. It simply prevents the contradiction from being "applied to produce arbitrary results," which would make "the application of mathematics into a farce."[57] But it does not explain the cause of the contradiction, nor does anything to remove it, therefore it does not contribute to reformulate flawed concepts. Had it been for Wittgenstein, the axioms of ZFC would never have been formulated.[58]

Since the above objections are unjustified, there seems to be no valid argument against the claim that mathematics cannot be said to be absolutely certain.

There also seems to be no valid argument against the claim that mathematical knowledge is not infallible, in the sense that it cannot be excluded that it might lead to error. In order to exclude this, one should at least be able to demonstrate the consistency of the axioms by absolutely reliable means. But, by Gödel's second incompleteness theorem, this is impossible.

Simpson states that today "the validity of mathematics is under siege."[59] In particular, "Gödel supplied heavy artillery for all would-be-assailants of mathematics."[60] Therefore, "mathematicians and philosophers of mathematics ought to get on with the task of defending their discipline."[61] The "attack on mathematics is part of a general assault against reason."[62]

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[54] Wittgenstein 1976, 209.

[55] *Ibid.*

[56] Wittgenstein 1978, III.80.

[57] *Ibid.*, VII.15.

[58] A brilliant parody of Wittgenstein's rule is implicit in Calvino's fictional dialogue between Marco Polo and Kublai Kahn; see Calvino 1974, 69.

[59] Simpson 1988, 357.

[60] *Ibid.*, 362.

[61] *Ibid.*, 358.

[62] *Ibid.*

Actually, almost the opposite is the case. What is against reason is to locate mathematics in an unlikely paradise of eternal, unrevisable truths, rather than taking it for what it actually is, that is, fallible, revisable, and evolving, like natural science.

That mathematics cannot be said to be absolutely certain is honestly acknowledged by Russell who, after decades of fruitless attempts to establish the absolute certainty of mathematics, states: “I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere.”[63] But, “having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering” and, “after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable.”[64] The “splendid certainty which I had always hoped to find in mathematics was lost in a bewildering maze.”[65]

7. *The Top-Down and Bottom-Up Approaches to Mathematics*

In the previous sections we have considered a number of objections against the use of Gödel’s incompleteness theorems to make claims about mathematics. A further objection is that Gödel’s incompleteness theorems are irrelevant to mathematical practice.

For example, Davies states: “I got the dread seeing yet another discussion of Gödel’s theorems and their importance, when I knew that they had almost no relevance to the work of most mathematicians.”[66]

In fact, after the discovery of Gödel’s incompleteness theorems, several mathematicians have continued to do mathematics as if Gödel’s results did not exist. Even their view of the nature of mathematics has been totally unaffected by Gödel’s results, since they have continued to believe that “the axiomatic method is ‘the’ method of mathematics, in fact, it is mathematics.”[67]

This is due to the fact that most mathematicians follow the top-down approach to mathematics, which has been the mathematics paradigm for the past one and a half century.

There are two approaches to mathematics: the top-down and the bottom-up approach.[68] According to the top-down approach:

1) A mathematics field is developed from above, that is, from general principles concerning that field.

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[63] Russell 1971, III, 220.

[64] *Ibid.*

[65] Russell 1995b, 157.

[66] Davies 2008, 88.

[67] Naylor and Sell 2000, 6.

[68] On the distinction between top-down and bottom-up approaches to mathematics, see Cellucci 2012.

2) It is developed by the axiomatic method, which is a downward way from principles to conclusions derived deductively from them.

Leibniz's approach to the calculus, as presented by de l'Hospital's in the first textbook on the subject, is an example of the top-down approach. Church's approach to computability in terms of the lambda calculus is another example.

That the top-down approach has been the mathematics paradigm for the past one and a half century is due to the influence of the Göttingen school, from Dirichlet, Riemann and Dedekind to Klein, Hilbert and Noether. It is also due to the influence of Bourbaki who, through Noether's student van der Waerden, was influenced by the Göttingen school.

But, by identifying the method of mathematics and science with the axiomatic method, the top-down approach is incompatible with Gödel's incompleteness theorems. And, by identifying mathematical reasoning with deductive reasoning, the top-down approach overlooks that "deduction plays only a small role in human reasoning."[69]

The alternative to the top-down approach is the bottom-up approach, according to which:

1) A mathematics field is developed from below, that is, from problems of that field or of other mathematics, natural science or social science fields.

2) It is developed by the analytic method, which is an upward way from problems to hypotheses derived non-deductively from them.

Newton's approach to the calculus of fluxions is an example of the bottom-up approach; see Chapter 20. Turing's approach to computability, which starts from of an analysis of the actions of a human being who is making a calculation, is another example.

While affecting the axiomatic method, Gödel's incompleteness theorems do not affect the analytic method. On the contrary, they provide evidence for it; see Chapter 16.

8. *The Top-Down and Bottom-Up Approaches to Science*

The top-down and bottom-up approaches extend from mathematics to natural and social science fields.

According to the top-down approach, a natural or social science field is developed from above, that is, from general principles concerning that field, and is developed by the axiomatic method.

For example, Dirac states that one has a "physical theory when all the axioms and rules of manipulation governing the mathematical quantities are specified and when in addition certain laws are laid down connecting physical facts with the mathematical formalism."[70] In an application of a physical theory "one would be given certain physical information, which one would proceed to express by equations between the mathematical

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[69] Oaksford and Chater 2007, 56.

[70] Dirac 1958, 16.

quantities. One would then deduce new equations with the help of the axioms and rules of manipulation."[71]

Conversely, according to the bottom-up approach, a natural or social science field is developed from below, that is, from problems of that field or of other natural or social science fields, and is developed by the analytic method.

For example, Grosholz states that "certain important areas of scientific activity have not been, and may never be, reformulated in terms of axiomatic theories."[72] And yet "they come to stand in deep and interesting relations to other areas of science in complex ways that contribute to scientific explanation," and "the derivation may fail to be deductive."[73] These areas of science are investigated by analysis, which is "the search for the conditions of solvability of a problem."[74] Such conditions are found by means of "non-deductive derivations," which "may themselves be powerful vehicles of scientific explanation."[75]

The top-down and bottom-up approaches correspond to two different views of the world.

The top-down approach corresponds to the view that all the facts of the world are to be explained in terms of some higher principle. In the theological version of this view, the higher principle is God, or something like God.

The top-down approach corresponds to the view that the world is to be explained in terms of physical processes, going from the more elementary to the more complex ones.

9. *Limitations of the Top-Down Approach*

In addition to being incompatible with Gödel's incompleteness theorems, the top-down approach has other limitations. Here are some of them as regards mathematics.

1) *The top-down approach does not permit to distinguish between mathematical theories in terms of their significance*. As Bourbaki acknowledges, being based on arbitrary axioms, subject only to the condition of consistency, the top-down approach led to "a whole crop of monster-structures, entirely without applications; their sole merit was that of showing the exact bearing of each axiom, by observing what happened if one omitted or changed it."[76] One might think that "these were the only results that could be expected from the axiomatic method."[77]

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[71] *Ibid.*
[72] Grosholz 2000, 81.
[73] *Ibid.*
[74] Grosholz 2007, 34.
[75] Grosholz 2000, 82.
[76] Bourbaki 1996, 1275, footnote 9.
[77] *Ibid.*

2) *The top-down approach leads to a fragmentation of the research.* As Bourbaki acknowledges, because of such an approach, “many mathematicians take up quarters in a corner of the domain of mathematics,” and “are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own.”[78]

3) *The top-down approach is not very successful in the application of mathematics to several fields.* Mathematics fields developed by means of it have not been very successful in applications to several natural or social science fields, from life sciences to economics. This is due to the fact that those mathematics fields have been developed from within, that is, from principles concerning their own basic concepts, rather than from without, that is, from the peculiarities of the natural or social science fields in question.

4) *The top-down approach is unable to explain the success of the application of mathematics to physics.* As Bourbaki acknowledges, such an approach makes us “completely ignorant as to the underlying reasons” for the “intimate connection between experimental phenomena and mathematical structures.”[79] It “so happens – without our knowing why – that certain aspects of empirical reality fit themselves into” mathematical structures “as if through a kind of preadaptation.”[80] One may ask “why do such applications ever succeed,” but, “fortunately for us, the mathematician does not feel called upon to answer such questions.”[81]

10. *Seeking Another Role for Mathematical Logic*

As a response to the failure of mathematical logic to give a secure foundation for mathematics, several people have sought another role for the subject, particularly in mathematics or in computer science.

This is unconvincing. On the one hand, the view of mathematical logic that mathematics is in essence deduction from axioms, does not explain why a demonstration yields something new; nor why, once a mathematician has found a solution of a mathematical problem, other mathematicians seek other solutions; nor why different mathematicians, with no contact with each other, solve the same mathematical problem independently.[82]

On the other hand, mathematical logic has had no crucial role in the most significant technological developments of real computing in the last decades. For example, large investments by Japan’s Ministry of International Trade and Industry in a Fifth Generation Computer Systems project, heavily dependent on mathematical logic, ended in failure and the project was eventually terminated.

This is no surprise because, as we have seen, the intended purpose of mathematical logic, the one for which it was created, was to give a secure foundation for mathematics. This shaped the subject and deeply affected its

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[78] *Ibid.*, 1266.
[79] *Ibid.*, 1275.
[80] *Ibid.*, 1276.
[81] Bourbaki 1949, 2.
[82] See Cellucci ?.

further development. It would have been extraordinary if mathematical logic had turned out to be fit to a completely different purpose. As we have seen, Frege himself stated that mathematical logic is a device invented for certain scientific purposes, and one must not condemn it because it is not suited to others.

11. *The Criticism of Scholastic Logic*

The failure of the attempts to seek another role for mathematical logic raises the question: What is logic, really? Is it merely the study of deductive reasoning, or has it a wider scope?

In the seventeenth century the then current logic paradigm, Scholastic logic, fell under the blows of Bacon, Galileo, Descartes, Locke.

Thus Bacon stated that Scholastic logic "is useless for the discovery of sciences."[83] It "is good rather for establishing and fixing errors (which are themselves based on vulgar notions) than for inquiring into truth; hence it is more harmful than useful."[84]

Galileo stated that Scholastic "logic teaches how to know whether or not reasonings and demonstrations already made and discovered are conclusive," but not "to find conclusive reasonings and demonstrations."[85]

Descartes stated that Scholastic logic "contributes nothing whatsoever to the knowledge of truth."[86] For it "does not teach the method by which something has been discovered."[87] Therefore, it "is entirely useless for those who wish to investigate the truth of things."[88]

Locke stated that Scholastic logic "discovers no new proofs," it is merely "the art of marshalling, and ranging the old ones we have already."[89] Therefore, it "has been thought more proper for the attaining victory in dispute, than for the discovery or confirmation of truth, in fair enquiries."[90]

Bacon, Galileo, Descartes, Locke highlighted a serious weakness of Scholastic logic: its inadequacy to the needs of modern science, being useless to discover anything new. At least in this respect, the present logic paradigm, mathematical logic, does not fare much better.

12. *Scholastic Logic and Mathematical Logic*

In fact, the condition of mathematical logic appears dangerously similar to that of Scholastic logic.

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[83] Bacon 1961–1986, I, 158.
[84] *Ibid.*
[85] Galilei 1968, VIII, 175.
[86] Descartes 1996, X, 406.
[87] *Ibid.*, VII, 156.
[88] *Ibid.*, X, 406.
[89] Locke 1975, 679.
[90] *Ibid.*, 677–678.

Admittedly, mathematical logic can account for inferences that Scholastic logic could not account for, such as: All horses are animals, therefore all heads of horses are head of animals. But the uselessness Bacon, Galileo, Descartes, Locke ascribed to Scholastic logic did not stem from a deficiency later removed by technical refinements. It was due to the fact that Scholastic logic was not an instrument of discovery, and this charge affects mathematical logic as well. It is untouched by the latter being a technical refinement over Scholastic logic.

To address the incapability of Scholastic logic to be an instrument of discovery, Bacon and Descartes tried to develop a logic of discovery. Since the effort was focused on that direction, in the seventeenth and eighteenth centuries there was a marked decline of interest in Scholastic logic.

In this connection, it is significant that historians of logic inspired by mathematical logic blame the Scientific Revolution for the decline of interest in logic during the seventeenth and eighteenth centuries, and consider logically irrelevant Bacon's and Descartes' attempts to develop a logic of discovery. In their view, the copious and passionate discussions on the scientific method during the Scientific Revolution were extrinsic to logic and even opposed to it.

Thus the Kneales state that "the rise of modern physics" and "the recognition that logic was not an instrument of discovery" led to "the marked decline of interest in formal logic which occurred during the seventeenth and eighteenth centuries."[91] Moreover, "logic was divorced at this time from mathematics."[92] Mathematicians focused their attention on "algebra and analysis" which "were not elaborated in axiomatic fashion."[93] They "came to think of their new methods as independent of traditional logic," since "the new branches of mathematics were supposed to contain a technique of discovery."[94]

Similarly, Blanché states that, "as a result of the scientific revolution carried out, most of all, by Galileo," a "radical transformation takes place: the sharp rejection of logic" and its replacement with "the making theoretically explicit of the method practiced by science."[95] The "best representative of this new attitude is Descartes," a philosopher who, like Bacon, "has given no real contribution to logic."[96] Indeed, in Bacon's and Descartes' work "there is strictly nothing to keep for the history of logic."[97]

13. *Mathematical Logic and Discovery*

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[91] Kneale and Kneale 1962, 307.

[92] *Ibid.*, 308.

[93] *Ibid.*, 309.

[94] *Ibid.*

[95] Blanché 1970, 175.

[96] *Ibid.*

[97] *Ibid.*, 174.

That, like Scholastic logic, mathematical logic is useless to discover anything new, is not an accident, but rather a consequence of Frege's assumption that the question of discovery is merely subjective.

According to Frege, "we can inquire, on the one hand, how we have gradually arrived at a given proposition and, on the other, how we can finally provide it with the most secure foundation."[98] The first question, namely discovery, is merely subjective, since it "may have to be answered differently for different persons," only the second question, namely justification, "is more definite."[99] Therefore, mathematical logic must concern itself "not with the way in which" mathematical propositions "are discovered but with the kind of ground on which their" justification "rests."[100] The "task of logic is to set up laws according to which a judgment is justified by others."[101]

Frege's view is shared by most mathematical logicians, according to whom, while deducing conclusions from given principles is a logical task, there is no logical way to scientific hypotheses, the latter can be reached only by intuition.

In the last century this has been the received view, but this does not mean that it is a sound one. Indeed, one of the aims of this book is to show that it is not.

14. *The Need for an Alternative Logic Paradigm*

Consistently with Frege's assumption that no logic of discovery is possible, mathematical logic has been useless to discover anything new. Frege himself states: "There are no new truths in my work."[102] This makes the condition of mathematical logic dangerously similar to that of Scholastic logic, and exposes it to the very same charge of uselessness that Bacon, Galileo, Descartes, Locke brought against Scholastic logic.

Not only mathematical logic has been useless to discover anything new, but it has not been significantly helpful even as a means of justification. A theorem cannot be more justified than the axioms from which it is deduced, and, by Gödel's second incompleteness theorem, no absolutely reliable justification for the axioms is generally possible.

As a result, mathematical logic has had little impact on scientific research. A discipline earns its keep if it proves capable of solving problems, first of all, those arising from the world around us and our participation in it. If it fails to do so, there is a sense of unreality about the whole discipline.

For such reason, Davis states that, although "it would be invidious to mention a specific example of exhaustion of a field when there are people working very happily in it," the "following example and opinion is in the

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[98] Frege 1967, 5.
[99] *Ibid.*
[100] Frege 1959, 23,
[101] Frege 1979, 175.
[102] Frege 1967, 6.

open literature. Classical mathematical logic" has "lost its connection to reality and has produced mathematical monsters."[103]

An alternative logic paradigm is necessary if logic is to play any significant role in scientific research.

Thus Rota states that "that magnificent clockwork mechanism that is mathematical logic is slowly grinding out the internal weaknesses of the system."[104] Its concepts "were invented one day for the purpose of dealing with a certain model of the world" which is "inadequate to the needs of the new sciences."[105] Today, "in all circumstances imaginable, including mathematical reflection (the true one, not the one of *a posteriori* reconstructions), logic shines for its absence."[106] If "we are to set the new sciences on firm, autonomous, formal foundations, then a drastic overhaul" of "logic is in order."[107]

An alternative logic paradigm is also necessary if logic is to play any significant role in philosophy.

Thus Heidegger states that, while logic is "quite correctly valued as an essential entry into philosophy," what today "is called logic," namely mathematical logic, "is not a logic at all and has nothing in common anymore with philosophy."[108] It "never becomes clear what use this logic is supposed to have," its "pursuit leaves the student outside philosophy, when it does not actually drive him from it."[109] Therefore, "there is need for another logic."[110]

Even some thoughtful mathematical logicians acknowledge that mathematical logic does not play any significant role in philosophy.

Thus Wang states that, "as we understand the nature of mathematical logic better, we find that the early belief in its philosophical relevance was largely an illusion."[111] Mathematical logic "is rather detached from actual knowledge."[112] It is "but a special branch of mathematics and, in fact, it is not often regarded as a very central branch"[113] Insofar as "it has been influential in philosophy, I am not sure the influences have been good ones."[114]

This lack of regard for mathematical logic contrasts with the consideration logic enjoyed in antiquity when, as Barnes points out, almost no one "disputed the idea that logic also supplied the sciences with its

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[103] Davis 2006, 176.
[104] Kac, Rota and Schwartz 1992, 180.
[105] *Ibid.*
[106] Rota 1999, 94.
[107] Kac, Rota and Schwartz 1992, 180.
[108] Heidegger 1984, 5.
[109] *Ibid.*
[110] *Ibid.*
[111] Wang 1974, 28.
[112] *Ibid.*
[113] *Ibid.*, 21.
[114] *Ibid.*, 51.

instruments."[115] For this reason, "in antiquity, logic was not an esoteric discipline, reserved" for "a few specialists. Rather, it was a standard part of the school curriculum."[116] Indeed, "every educated man was soaked in logic," and "logic was an uncontested basis for the study of science and philosophy."[117]

15. *Toward an Alternative Logic Paradigm*

Toward an alternative logic paradigm, this book examines the limitations of mathematical logic and outlines a new approach to logic. This is aimed at giving a more satisfactory answer to the question: What is logic, really?

This motivates the subtitle of the book, which is reminiscent of Hersh's *What Is Mathematics, Really?* Hersh describes his book as "a subversive attack on traditional philosophies of mathematics," which are a "relic of the Frege-Russell-Brouwer-Hilbert foundationist philosophies."[118] Without pretending to be as happily subversive as Hersh's book, this book seeks to lay the basis for an approach to logic more adequate to today's needs. In view of this, another appropriate subtitle for the book would be: What is logic today, and what should it be?

In order to see what logic should be, it is illuminating to compare mathematical logic with earlier views of logic and its relation with the scientific method.

To this purpose, the book gives an overview of how logic and its relation with the scientific method have been conceived in antiquity and in the modern age. But, while conveying some sense of the unfolding story, this book is by no means a history of logic. It only focuses on certain landmarks in the views on logic and its relation with the scientific method, which may provide useful indications for a new approach to the subject. This explains why, for example, Part I of the book ends with the Stoics' deductivist turn and Part II begins with Galileo's method, with nothing in between.

From the overview in question it appears, in particular, that, contrary to a widespread opinion, mathematical logic is a substantial restriction on the scope of logic with respect to Plato, Aristotle, Descartes and Kant.

Since one of the most serious limitations of mathematical logic is that it is useless to discover anything new, in the alternative logic paradigm logic is intended, first of all, to provide rules of discovery, that is, non-deductive rules for finding hypotheses to solve problems. While mathematical logic, being a logic of justification, is only aimed at systematizing and justifying what is already known – and, by Gödel's incompleteness theorems, it fails even to do that – in the alternative logic paradigm logic is aimed at providing means to discover something new, so it is intended to be, first of all, a logic of discovery. This is essential if logic is to play any relevant role in mathematics, science and even philosophy.

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[115] Barnes 2003, 22.
[116] *Ibid.*
[117] *Ibid.*, 23.
[118] Hersh 1997, xi–xii.

16. *Characters of the Alternative Logic Paradigm*

The alternative logic paradigm outlined in this book is not merely an expansion of mathematical logic. It has different presuppositions, it uses different rules and develops aspects that previously received little notice. In particular, it involves a new view of the relation of logic with evolution, language, reason, method and knowledge.

1) *Evolution*. According to mathematical logic, logic is totally independent of biological evolution. According to the alternative logic paradigm, logic is a continuation of the processes for solving problems with which biological evolution has endowed all organisms.

2) *Language*. According to mathematical logic, logic is essentially dependent on language, since it is only concerned with propositional inferences, that is, inferences from propositions to propositions. According to the alternative logic paradigm, logic is also concerned with non-propositional inferences.

3) *Reason*. According to mathematical logic, logic is the whole of reason. According to the alternative logic paradigm, logic is only a part of reason.

4) *Method*. According to mathematical logic, logic does not originate from method. According to the alternative logic paradigm, method is the source of logic.

5) *Knowledge*. According to mathematical logic, logic does not provide means to acquire knowledge. According to the alternative logic paradigm, logic provides such means.

Such differences between the alternative logic paradigm and mathematical logic will be discussed in Chapters 15 and 16.

17. *The Alternative Logic Paradigm and Philosophy*

The alternative logic paradigm also involves a new view of the relation of philosophy with knowledge.

According to mathematical logic, and analytic philosophy which is strictly related to it, philosophy does not provide means to acquire knowledge, it only clarifies what we already know.[119] According to the alternative logic paradigm, philosophy may provide such means.

This extends the relation of logic with knowledge to the whole of philosophy.

The differences between the alternative logic paradigm, on the one hand, and mathematical logic and analytic philosophy, on the other hand, will be discussed in Chapter 18.

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[119] Here and in what follows, ‘analytic philosophy’ means ‘mainstream analytic philosophy’, a tradition including Cambridge philosophers, such as Russell, Moore, Wittgenstein; Vienna Circle philosophers, such as Schlick, Carnap, Neurath; Berlin philosophers, such as Reichenbach, Hempel; Oxford philosophers, such as Ryle, Austin, Strawson, Dummett.

18. *The Reconstruction of Logic*

About a century ago Dewey published a book entitled *Reconstruction in Philosophy*. Some twenty-five years later, in a new Introduction to the book, he stated: "Today Reconstruction *of* Philosophy is a more suitable title than Reconstruction *in* Philosophy."[120] The reconstruction of philosophy in particular involved a reconstruction of logic, since "contemporary logical theory is the ground upon which all philosophical differences and disputes are gathered together and focussed."[121]

Dewey's problem is still with us. A reconstruction of philosophy and logic is necessary if they are not to become an anachronism. The alternative logic paradigm outlined in this book aspires to suggest a way toward such a reconstruction.

In view of its substantial differences from mathematical logic, the alternative logic paradigm is likely to meet opposition from mathematical logicians.

They, however, should consider that, as van Benthem says, "we cannot keep singing hymns to Gödel and Tarski forever – unless we are already in Heaven."[122] Indeed, "I have never been able to understand this defense of the status quo. In particular, I have always found many outspoken critics of modern logic" to be "well-worth reading, and a useful reminder of the many doors our Founding Fathers have closed historically – doors that could be opened again now."[123]

Such doors not only could, but should be opened again now because, as it has been mentioned above, mathematical logic is only aimed at systematizing and justifying what is already known, and, by Gödel's incompleteness theorems, it fails even to do that.

19. *Organization of the Book*

The book is divided into four parts: Ancient Perspectives, Modern Perspectives, An Alternative Perspective, Rules of Discovery.

Part I, 'Ancient Perspectives', deals with some basic perspectives on logic in antiquity.

Chapter 1 deals with the origin of logic and its relation with method, as well as with the origin of the names 'logic' and 'method'. Chapter 2 deals with Plato's and Aristotle's conceptions of science, which involve two different methods, the analytic method and Aristotle's analytic-synthetic method, respectively. Chapter 3 deals with the analytic method. Chapter 4 deals with Aristotle's analytic-synthetic method and Pappus' variant of it. Chapter 5 deals with the deductivist view of Aristotle's logic, which interprets it as the study of deduction. Chapter 6 deals with the heuristic view

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[120] Dewey 2004, iii.
[121] *Ibid.*, 77.
[122] van Benthem 2008, 40.
[123] *Ibid.*

of Aristotle's logic, which interprets it as primarily a logic of discovery. It is argued that only this view is adequate.

Part II, 'Modern Perspectives', deals with some basic perspectives on logic in the modern age.

Chapter 7 deals with the method of modern science, as stated by Galileo and Newton, showing that it is essentially the same as Aristotle's analytic-synthetic method. Chapter 8 deals with Descartes', Leibniz's and Kant's conceptions of logic and, in particular, with their views on the logic of discovery. Chapter 9 deals with Frege's conception of logic, in particular with Frege's ideal of splitting up deduction into a few logically simple modes of inference, and his failure in achieving this ideal. Chapter 10 deals with Gentzen's analysis of deduction and his failure in achieving this very same ideal. Chapter 11 deals with the impact of Gödel's incompleteness theorems and other limitative results on the tenets of mathematical logic. Chapter 12 deals with the divorce of logic from method operated by Frege, and with the limitations of the psychology of discovery which originated from this divorce.

Part III, 'An Alternative Perspective,' outlines a perspective on logic hopefully not subject to the limitations of mathematical logic.

Chapter 13 deals with the nature of reason and knowledge. Chapter 14 deals with the relation of reason and knowledge with emotion. Chapter 15 deals with the relation of logic with evolution, language and reason. Chapter 16 deals with the relation of logic with method and knowledge. Chapter 17 deals with the question of classification and justification of inference rules, both deductive and non-deductive. Chapter 18 deals with the relation of philosophy with knowledge.

Part IV, 'Rules of Discovery', deals with the rules of discovery, namely, non-deductive rules for finding hypotheses to solve problems.

Chapter 19 deals with induction and analogy, indeed, various kinds of them. Chapter 20 deals with generalization, specialization, metaphor, metonymy, definition as abbreviation, definition as analysis, and diagrams.

The book is the result of a constant struggle between scope and depth. There is scarcely a chapter that would not be worthy of further expansion. But, as Sterne says, "no author, who understands the just boundaries of decorum and good-breeding, would presume to think all: The truest respect which you can pay to the reader's understanding, is to halve this matter amicably, and leave him something to imagine, in his turn, as well as yourself."[124]

20. *A Terminological Remark*

Throughout the book the term 'demonstration' is used instead of 'proof', because today 'proof' is commonly used to indicate axiomatic demonstration.

The latter is at the core of mathematical logic. In view of the limitations of the latter, in the alternative logic paradigm proposed in this book, the axiomatic notion of demonstration is replaced with the analytic one.

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[124] Sterne 1997, 88.

To avoid misunderstandings, it seems therefore preferable to use ‘demonstration’ instead of ‘proof’.

21. *Notations, Quotations, Transliterations*

The following notations are used for logical constants: $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if...then), $\leftrightarrow$ (if and only if), $\forall$ (for all), $\exists$ (for some, there exists).

When quoting from ancient Greek philosophers, and even from some modern ones, translations are original unless otherwise stated. This is motivated by the fact that, first, every translation is an interpretation, and the interpretations proposed in this book are often different from those on which current translations are based; secondly, current translations of different works by different translators may be inconsistent with each other, so quoting from them would lead to misunderstandings.

When quoting Greek expressions, the so-called scientific transliteration from the Greek to the Latin alphabet is used.

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