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PENELOPE MADDY, *Realism in mathematics*, Oxford (Clarendon Press) 1990; revised 1992; *ix*+204 pp.

As a result of the failure of the foundational programs, in the past three decades there has been a revival in the empiricist philosophy of mathematics. In the latter several trends can be distinguished and among them an eminent position is occupied by a Millian one. A distinctive feature of the supporters of such trend is that, while considering the absolute certainty of mathematical propositions as an illusion, they do not intend to suggest skepticism as to the reliability of logical deduction. Like Mill they consider logical inference as absolutely infallible, only axioms are fallible. A significant representative of such trend is Penelope Maddy whose views, first proposed in a series of articles, are systematically presented in this book (RM for short).

Classifying Maddy as a Millian may seem peculiar because her scanty remarks about Mill in RM are fairly critical. For example, in rejecting the view that our most basic general set theoretic beliefs are justified by simple enumerative induction, she claims that: "the idea that mathematics is a simple inductive science goes back to Mill" (RM, p. 67, footnote 77). However, including Maddy in the Millian camp seems justified in view of the following facts.

1) Like Mill, Maddy holds that it is impossible to ascribe any character of necessity to the truths of mathematics and to assume "that they are, in any sense, infallible" (RM, p. 71). They "depend on the empirical facts of current science", and so are "a posteriori and fallible" (Maddy 1992, p. 285). In particular "set theory is as fallible as any other science" (RM, p. 128, footnote 62). While set theory is the foundation of mathematics, "such a foundation does not provide epistemic certainty, only ontological unity" (Maddy 199?). Today "no one would expect even the best scientific arguments to be absolutely justifying" and our epistemological inquiries in mathematics might even "be hampered if we set an unreasonably high standard" (Maddy 1988, p. 761).

2) Like Mill, Maddy claims that the mathematical method must be identified with the axiomatic method because "proof must begin from axioms that are not themselves proved" (RM, p. 144). She maintains that theorems "are justified by deductive inference, and that this is a reliable inferential process" (Maddy 1984, p. 49). Indeed, all of mathematics can be developed within a single axiomatic system, i.e. axiomatic set theory. In contemporary mathematics, "a question about what sorts of mathematical things or

structures exist is ultimately a question about what sorts of sets exist, and a question about what can be proved is ultimately a question about what can be proved from the axioms of set theory". The latter provide "a framework in which all classical mathematical objects and structures can be defined and all classical mathematical theorems proved". Conversely, giving "a foundation in this sense is one of the goals of contemporary set theory". For, one of its aims "is to provide sets that will realize all mathematical structures" and hence "to provide an ontological foundation for mathematics" (Maddy 199?).

3) Like Mill, Maddy believes that, since the mathematical method consists of the axiomatic method and theorems are justified by deductive inference, "all that's left is accounting for our knowledge of the axioms" (Maddy 1984, p. 49). According to the axiomatic method, proof "must begin from axioms that are not themselves proved" (RM, p. 144). They cannot be proved and must be justified by some non-demonstrative argument. Therefore we must describe "what we take to be a reliable mechanism which produces our belief in the mathematical axioms" (Maddy 1984, p. 50). An account of our "knowledge of axioms and of the evidential role of non-demonstrative mathematical arguments in general is clearly needed" (RM, p. 144). In particular, in view of the role of set theory in providing a foundation for mathematics, understanding how the axioms of set theory are justified and thus developing "a suitable methodology for set theoretic axiomatics, looms as the most pressing foundational problem in contemporary mathematics" (Maddy 1991, p. 159).

4) Like Mill, Maddy maintains that in order to describe what we take to be a reliable mechanism which produces our belief in at least some mathematical axioms we must ultimately rely on intuition. For Mill "we never could arrive at any knowledge by reasoning, unless something could be known antecedently to all reasoning", so there must be certain elementary beliefs which "are known directly, and of themselves". Such beliefs "are the subject of Intuition" and "are the original premises from which all others are inferred". They cannot be object of logic because "with the original data, or ultimate premises of our knowledge; with their number or nature, the mode in which they are obtained, or the tests by which they may be distinguished", logic has "nothing to do" (Mill 1973-74, pp. 7-8). Similarly for Maddy mathematical knowledge depends on some "primitive general beliefs about physical objects that are not supported by simple enumerative induction" (RM, p. 68) and must ultimately rely on intuition.

5) Like Mill, Maddy maintains that "the only form of inferential justification available for the mathematical axioms is inductive" (Maddy 1984, p. 51). While it is true that "simple enumerative induction is not suitable for mathematics", one may adopt "a more theoretical form of induction" which "yields a position like Quine's or Putnam's"

(Maddy 1984, p. 51). The latter is summarized by the following ‘indispensability argument’: “We are committed to the existence of mathematical objects because they are indispensable to our best theory of the world and we accept that theory” (RM, p. 30). Similarly Maddy holds that, since mathematical methods “have effectively produced all of mathematics, including the part so far applied in physical science”, it is by them that we can “best determine precisely what mathematical things there are and what properties these things enjoy” (Maddy 1992, pp. 279-80). For example, the fact that the calculus is indispensable in physics and the set-theoretic continuum provides the best account of it, “justifies our belief in the set-theoretic continuum, and so, in the set-theoretic methods that generate it” (Maddy 1992, p. 280). Thus we are entitled to assume that “our best theory of mathematical ontology is that (at least some) mathematical entities are sets” (RM, p. 59).

6) Like Mill, Maddy is aware that, although the only form of inferential justification available for the mathematical axioms is inductive, the indispensability argument does not account for the whole of mathematics. For example, it does not account for unapplied mathematics which is “completely without justification on the Quine/Putnam model” (RM, p. 30). Moreover, as pointed out by Parsons, it “leaves unaccounted for precisely the obviousness of elementary mathematics” (Parsons 1979/80, p. 151). In view of this Maddy proposes a “compromise between Quine/Putnam and Gödelian Platonism”, according to which “successful applications of mathematics give us reason to believe that mathematics is a science” and justify “the practice of mathematics”, but in order to account for unapplied mathematics and for the obviousness of elementary . mathematics one must appeal to intuition. This means that “we need to explain what intuition is and how it works” (RM, pp. 34-35). According to Maddy such an explanation is given by Hebb’s analysis of neural operations which suggests that the development of higher-order cell-assemblies responsive to particular sets of physical objects “gives rise to an even higher-order assembly corresponding to the general concept of set”. The structure of this general set assembly is responsible for many of our “intuitive beliefs about sets, for example that they have various subsets, that they can be combined, and so on”. Such intuitive beliefs “underlie the most basic axioms of our scientific theory of sets” (RM, p. 70), for example Pairing.

7) Like Mill, Maddy has a two-tiered conception of scientific knowledge: the most primitive truths are given by intuition while the more theoretical hypotheses are justified by their consequences. According to Mill the more theoretical principles, which are “assumed as premises for the purpose of deducing from them the known laws of concrete phenomena” (Mill 1973-4, p. 483), are justified “under the idea that if the conclusions to

which the hypothesis leads are known truths, the hypothesis itself either must be, or at least is likely to be, true" (Mill 1973-4, p. 490). Similarly for Maddy only the most elementary axioms of set theory can be justified in terms of intuition, but "the more theoretical hypotheses are justified extrinsically, by their consequences, by their ability to systematize and explain lower-level theory and so on" (RM, p. 107). In the case of set-theoretic axioms such an extrinsic evidence comes in a variety of forms, like having verifiable consequences, providing powerful new methods for solving pre-existing open problems, providing proofs of statements previously conjectured, implying 'natural' results, providing new proofs of old theorems, filling a gap in a previously conjectured false but natural proof, unifying new results with old so that the old become special cases of the new ones, providing strong intertheoretic connections, extending patterns begun in weaker theories. In particular the support for "the assumption of an infinite stage is purely extrinsic, following from the immense success of modern infinitary mathematics" (RM, p. 141). All such forms of extrinsic evidence "more or less correspond to forms of confirmation recognized in the physical sciences" (Maddy 1988, p. 759).

8) Like Mill, Maddy focuses her attention on the nature of mathematical objects, assuming that they do not exist apart from the material world but only as embodied in it. She wants to "bring them into the world we know and into contact with our familiar cognitive apparatus" (RM, p. 48). Thus her position is Aristotelian insofar as "Aristotle's forms depend on physical instantiations" (RM, p. 158). For her, "every physical thing is already mathematical, and every mathematical thing is based in the physical" (RM, p. 157). Maddy envisages "a spatio-temporal reality inseparably physical and mathematical" (RM, p. 158). From this viewpoint "sets no longer count as 'abstract' " (RM, p. 58). They have location in space and time; for example a set of three apples "is located exactly where they are" (RM, p. 59). Therefore sets are objects of ordinary sense-perception. Just like we perceive physical objects, so "we can and do perceive sets" (RM, p. 58). For example, when we see three apples in a box we perceive a set of three apples. Of course the world does not consist only of physical objects and sets of physical objects but also of sets of such sets "and so on, through the transfinite levels of the iterative hierarchy" (RM, p. 128). Now, a set of higher order will "again be located where its members are" (RM, p. 59). Even an extremely complicated set will "have spatio-temporal location, as long as it has physical things in its transitive closure" (RM, p. 59). Each set, "no matter how exalted in rank, is located where the physical stuff in its transitive closure is located" (RM, p. 156). Thus the subject matter of set theory consists of the "hierarchy generated from the set of physical individuals by the usual power set operation" (RM, p. 156).

From this summary of Maddy's position it appears that it broadly agrees with Mill's view as regards the nature of the mathematical method, of mathematical objects and of the certainty of mathematical propositions. Does it provide a plausible picture of the nature of mathematics? It raises a number of problems.

1) *The indispensability argument*. As Maddy has recently acknowledged, one cannot say that, since the calculus is indispensable in physics and the set-theoretic continuum provides the best account of the calculus, our belief in set theory is justified. To say that would mean to overlook that "indispensability for scientific theorizing does not always imply truth" (Maddy 1992, p. 289). Many applications of mathematics to physical science depend on assumptions that are literally false. For example, in fluid dynamics we assume that liquids are continuous. Although such a false assumption is indispensable to make the theory workable, we know perfectly well that liquids consist of molecules and that the assumption of continuity is only an approximation. Moreover, while from the viewpoint of the indispensability argument the choice between alternative axioms, say, in set theory, should hinge on developments in physics, "set theorists do not regularly keep an eye on developments in fundamental physics" and there seems to be "no mathematical reason to criticize this practice". So "legitimate choice of method in the foundations of set theory does not seem to depend on physical facts in the way indispensability theory requires" (Maddy 1992, p. 289).

Maddy recognizes that, if these objections could be sustained, then one should conclude that "the indispensability arguments do not provide a satisfactory approach to the ontology or the epistemology of mathematics", and that, given the centrality of the indispensability considerations in her position, this would require "a significant reorientation in contemporary philosophy of mathematics" (Maddy 1992, p. 289). She also acknowledges that "from a naturalist's perspective, the role of mathematics in science and the implications of that role for the foundations of set theory are more complex and subtle than has heretofore been appreciated" (Maddy 1994, p. 407). However Maddy provides no suggestion as to such a reorientation, probably because this would involve a major change in her position, in particular would require giving up the assumption that the axiomatic method is the method of mathematics.

2) *The two-tiered conception*. The proposed extrinsic criteria about current set-theoretic axiom candidates (verifiable consequences, symplifying and systematizing, etc.) not only do not allow to decide which (if any) of such candidates is true but do not even allow to assert that speaking of their truth makes sense at all. Most people working in set theory think otherwise. They seem resigned to the idea that, because of the impossibility

of deciding the truth of alternative set-theoretic axiom candidates, “mathematics is inevitably going to fragment, and protests about ‘unity of mathematics’ should be treated with the sort of attitude we take to the ‘unity of science’ “ (Drake 1985, p. 35). In any case the extrinsic criteria are not sufficient to justify any new set-theoretic axiom candidate because, according to Maddy, in addition to satisfying them, any such candidate would have to be consistent with the existing axioms. Indeed, not only “one of the central motivations behind axiomatization in the first place was to avoid the inconsistencies of naive set theory”, but “we must grant that the goal of avoiding inconsistent theories is fundamental to contemporary set theory” (Maddy 199?). But by Gödel’s second incompleteness theorem, if any of these axiom candidates “is in fact consistent, we shall never have a mathematical proof of that on the basis of the mathematics known today” (Drake 1985, p. 29).

3) *The need for rules of thumb*. Maddy agrees that even her two-tiered approach is not sufficient to account for the axioms of set theory. There are various arguments concerning large cardinal axioms “that are not happily classified as either intuitive or extrinsic” (RM, p. 140). Such arguments, which Maddy calls ‘rules of thumb’, do not depend on simple intuition because “they extend beyond anything that could plausibly be traced to an underlying perceptual, neurological foundation”. On the other hand they cannot be counted as extrinsic because they are only “indirectly subject to extrinsic support - if they consistently led to ineffective theories, they would eventually be dropped - but this proves nothing” (RM, p. 141). Now, Maddy provides no plausible justification for such arguments, limiting herself to say that a “central aspect of the appeal of these rules of thumb” is “that they ‘seem right’ ” (RM, pp. 141-2). This could hardly be regarded as a satisfactory foundation for them.

. 4) *The physical reality of sets*. It seems implausible to say that we do perceive sets because they exist in the material world. What we perceive are aggregates of physical stuff, not sets. We perceive three apples, not a set of three apples which is a different entity from the aggregate of its members. Moreover, to make plausible the idea that sets exists in the material world, one would have to show that infinite sets are physically real and the only way to do that would be to show that the continuous intervals of points are physically real. Now, this is considered problematical by the physical community. For example, according to Feynman, “the theory that space is continuos is wrong, because we get these infinities and other difficulties”; in particular “the simple ideas of geometry, extended into infinitely small space, are wrong” (Feynman 1967, p. 166). According to Isham, “the construction of a ‘real’ number from integers and fractions is a very abstract mathematical procedure, and there is no a priori reason why it should be reflected in the empirical world. Indeed, from the viewpoint of quantum theory, the idea of spacetime points seems singularly inappropriate" (Isham 1989, p. 72). Wheeler asks: "The spacetime continuum? Even continuum existence itself? Except as idealization neither the one entity nor the other can make any claim to be a primordial category in the description of nature" (Wheeler 1990, p. 138).

Of course one could always maintain that regarding infinite sets as physically real only means that they are used in the mathematical part of our best confirmed physical theory, in other words, that we must consider as physically real those entities which are presupposed by such a theory. But then the question falls back on the indispensability argument whose plausibility, as we have seen, is dubious.

5) *The feeling of infallibility*. How does it come about that certain hardly intuitive mathematical statements, such as the Chinese remainder theorem as stated in algebraic number theory, confer such a strong sense of being infallible, of involving notions so clearly understood as to raise no doubt about their certainty? Failing a convincing answer to this question Maddy's stress on the fallibility of mathematics can be challenged, not so much for being unrealistic as for being refuted by straightforward counterexamples.

Maddy seems unable to provide an answer to this problem. Her neurologically based picture of the process which produces our basic intuitive beliefs about sets does not justify their infallibility. A source of potential error is "the uncertain transition from intuitive belief to linguistic formulation", another one is "the distinct possibility that the intuitive belief itself is false". Thus, "in scientific contexts, intuitive beliefs must be tested like any other hypothesis, and like any other hypothesis, they can be overthrown" (RM, p. 71). It is true that the strength of our conviction that, say, the axiom of "Pairing is obviously true, along with the prevalence of similar convictions in others, supports the claim that Pairing is a good linguistic rendering of an intuitive belief, and the fact that a belief is intuitive lends prima facie support to the claim that it is true". Our belief in the Pairing axiom is corroborated if "further theoretical support is forthcoming, for example evidence that the axiom is consistent, that it produces theorems of the sort expected, and so on" (RM, p. 73). However, the fact remains that "we might be radically mistaken in the concepts we form". For example, it is possible that "sets actually don't have subsets, hard as it is for us to imagine such things". Indeed "some intuitive beliefs about sets have in fact been falsified by the progress of science", such as Frege's belief that "every property determines a set of things with that property" (RM, p. 71).

6) *The role of set theory in mathematics*. For Maddy a question about what sorts of mathematical structures exist is ultimately a question about what sorts of sets exist, and a question about what can be proved in mathematics is ultimately a question about what can be proved from the axioms of set theory. Now, most mathematicians would dispute that. Set theory is generally considered as somewhat remote by the working mathematicians who "could hardly see direct, relevant applications to their work of set theory" (Longo 1991, p. 120).

Assuming that a question about what can be proved in mathematics is ultimately a question about what can be proved from the axioms of set theory is purely ideological and not useful in practice. For example, when number theory is reduced to set theory we do not change our manner of doing it. Even if there is a sense in which our numerical calculations and number-theoretical proofs could be translated into set theory, doing number theory remains quite different from making calculations and proving number-theoretic results in set theory: it would be ridiculous to make calculations using the encoding of natural numbers by sets. Moreover, there is a good reason for proving number-theoretic results in number-theory instead of deriving them from the axioms of set theory via such an encoding: proofs are significantly less complex, not merely in the sense of containing less symbols but in the more basic sense of being more easily grasped.

Moreover, Maddy's view does not explain why, among all possible consequences of the axioms of set theory, one should actually choose those that are used in current mathematical practice, or why we find certain results more interesting than others. Actually, from Maddy's viewpoint only the axioms of set theory matter. All mathematical theorems are implicitly contained in them because they are ultimately obtained by repeated applications of fairly elementary logical rules. Indeed there is an absolutely trivial mechanical procedure that, given sufficient time and space, would generate all theorems using such rules, the so-called British Museum algorithm.

7) *Mathematical discovery*. Maddy sharply distinguishes between the context of justification and the context of discovery and considers impossible a logic of mathematical discovery. In her opinion the formation and justification of our most elementary beliefs is an object of neurology rather than of logic, and the formation and justification of our more advanced beliefs about sets is non-logical insofar as it involves some "nondemonstrative set theoretic arguments" (Maddy 1988, p. 762). Like Frege, Maddy puts off the question of discovery and is only concerned that not enough attention is given to the "question of when and why the assumption of various axioms is justified" (Maddy 1988, p. 481). In her view this problem is "the deepest that contemporary mathematics presents to the contemporary

philosopher of mathematics" (Maddy 1988, p. 482). The question of axiom justification "is especially pressing in current set theory, where the search is on for new axioms to determine the size of the continuum" (Maddy 1988, p. 482). There is an obvious similarity between such question and "the central business of philosophers of science: giving a confirmation of scientific theories" (RM, p. 146).

According to Maddy, the similarity is particularly clear in the case of the axioms of set theory, where it is apparent from "the very description of the styles of extrinsic justification - verifiable consequences, simplifying and systematizing theory, strong intertheoretic connections" (RM, p 146). What is needed is not just a description of such styles but "an account of why and when they are reliable, an account that should help set theorists make a rational choice between competing axiom candidates" (RM, p. 148). Now, such an account is not provided by Maddy: "Filling in the details of structure of non-demonstrative, non-intuitive arguments and evaluating their cogency is a subject for another book, a book I unfortunately don't know how to write" (RM, p. 146). She does not seem aware that clarifying the structure of non-demonstrative, non-intuitive arguments requires a logic of mathematical discovery.

From the above problems with Maddy's position it appears that, while mathematical empiricism has brought a breath of fresh air in the otherwise somewhat stagnating atmophere of contemporary philosophy of mathematics, the problems it raises are so serious as to suggest that we must make a fresh start. Maddy criticizes Gödel for presenting a two-tiered account of justification within mathematics which "fails to support the scientific status of mathematics as a whole and rests its account of elementary knowledge on an unpersuasive notion of mathematical intuition" (RM, pp. 177-178). I am afraid that much the same can be said of her own approach. Nevertheless her book deserves credit for exploring the potentialities of mathematical empiricism and showing its limitations.

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