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Mathematical Discourse vs. Mathematical Intuition

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1. Axiomatic Method and Theology

One of the most uninformative statements one could possibly make about mathematics is that the axiomatic method expresses the real nature of mathematics, i.e., that mathematics consists in the deduction of conclusions from given axioms. For the same could be said about several other subjects, for example, about theology. Think of the first part of Spinoza's *Ethica ordine geometrico demonstrata* or of Gödel's proof of the existence of God, which are both fine specimens of *Theologia ordine geometrico demonstrata*.

To the objection, 'Surely theological entities are not mathematical objects', one could answer: How do you know? If mathematics consists in the deduction of conclusions from given axioms, then mathematical objects are given by the axioms. So, if theological entities satisfy the axioms, why should not they be considered mathematical objects?

Hilbert says: "If in speaking of my points", lines and planes "I think of some system of things, e.g. the system: love, law, chimney sweep ... and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras' theorem, are also valid for these things".[1] Similarly he might have said: If in speaking of my points, lines and planes, I think of a suitable triad of theological entities, and assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras' theorem, are also valid for these things.

Indeed, if mathematics consists in the deduction of conclusions from given axioms, then it has no specific content. So it is simply impossible to distinguish geometrical objects, such as 'points, lines and planes', from 'love, law, chimney sweep', or a suitable triad of theological entities. This is vividly illustrated by Russell's statement that "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true".[2]

[1] Hilbert 1980, p. 40.
[2] Russell 1994, p. 76.

Hilbert's answer to the objection that, if mathematics consists in the deduction of conclusions from given axioms, then it has no specific content, is that the "circumstance" just "mentioned can never be a defect in a theory, and it is in any case unavoidable".[3] Moreover, "it takes a very large amount of ill will to want to apply" the axioms of geometry to other things "than the ones for which they were meant".[4] For applying them "always requires a certain measure of good will and tactfulness".[5]

Such an answer, however, is inadequate, because appealing to good will and tactfulness is not part of the axiomatic method, and in any case does not provide any specific content for mathematics.

Another answer to the objection that, if mathematics consists in the deduction of conclusions from given axioms, then it has no specific content, is provided by Hintikka. He claims that, contrary to Russell's statement, "we can hope to reach a point where we do know what we are talking about in mathematics, in the sense of being able to formulate descriptively complete theories for different mathematical theories".[6] By 'descriptively complete theories' Hintikka means theories whose models "comprise only the intended models. If there is only one intended model (*modulo* isomorphism), descriptive completeness means categoricity".[7]

Examples of such theories are provided by certain second-order theories, i.e., theories based on second-order logic. Of course, second-order logic is incomplete, in the strong sense that there is no consistent recursive set of rules such that every second-order consequence of second-order axioms can be deduced from such axioms by means of such rules. But, according to Hintikka, this is not in conflict with the view that mathematics consists in the deduction of conclusions from given axioms. The incompleteness of second-order logic does not "force us to search for new axioms, for the old ones", being descriptively complete, "already imply everything. What is needed are stronger and stronger formal rules of logical inference, calculated to capture more and more of the model-theoretical consequence relations".[8] Thus "the true novelties are better logical proof methods, not new axioms. In a sense, this would vindicate the idea of mathematics as being concerned primarily with proving theorems from axioms", except that "the proof would not rely exclusively on a closed list of rules of inference but might involve the discovery of new valid rules of inference".[9]

Such an answer, however, is inadequate. As Hintikka himself stresses, in the axiomatic method axioms "are supposed to tell you everything there is to be told" about a given subject matter, and "the rest of your work will consist in merely teasing out the logical consequences

[3] Hilbert 1980, p. 41.
[4] *Ibid*.
[5] *Ibid*.
[6] Hintikka 2000, p. 44.
[7] Hintikka 1996, p. 91.
[8] Hintikka 2000, p. 44.
[9] *Ibid*.

of the axioms. You do not any longer need any new observations, experiments or other inputs from reality. It suffices to study the axioms".[10] Now, since second-order logic is incomplete, proving a theorem from given second-order axioms might involve the discovery of new valid rules of inference, which in turn might involve the discovery of new axioms. So proving a theorem from second-order axioms might involve the discovery of new axioms. Then the given second-order axioms do not suffice, you need new observations, experiments or other inputs from reality, external to the given second-order axioms, to discover new axioms. That does not fit in with the axiomatic method.

Even Hintikka acknowledges that, "in practice, such stronger aids of deduction" can "often be codified in the form of new axioms for the mathematical theory in question", so the task of finding such stronger aids of deduction "is not entirely unlike the task of finding stronger and stronger axioms of set theory".[11] But this is hardly compatible with his claim that the fact that the true novelties are better logical proof methods would vindicate the idea of mathematics as being concerned primarily with proving theorems from given axioms. Hintikka goes as far as saying that, "contrary to the oversimplified picture that most philosophers have of mathematical practice, much of what a mathematician actually does is not to derive theorems from axioms".[12] Quite so.

2. Axiomatic Method and Intuition

Even granting, for argument's sake, that mathematics consists in the deduction of conclusions from given axioms, the question arises: How are axioms justified?

The view that mathematics consists in the deduction of conclusions from given axioms is incapable of providing a satisfactory answer to such crucial question. For axioms, being primary, cannot be proved by discourse, and so any justification of them must appeal to some non-discursive source of knowledge. Now, traditionally, by 'non-discursive source of knowledge', some kind of intuition is understood. But all known justifications of axioms in terms of intuition are inadequate.

This applies in particular to the two main such justifications, which are due to Gödel and Hilbert, respectively.

Gödel claims that axioms are justified if they are based on mathematical intuition. By that he means intellectual intuition – what Kant, though denying that this kind of intuition is humanly possible, calls "pure intuition, intellectual and exempt from the laws of the senses".[13] According to Gödel, "mathematical objects and facts", specifically those

[10] Hintikka 1996, p. 1.
[11] *Ibid*., p. 99.
[12] *Ibid*., p. 95.
[13] Kant 1900-, II, p. 413.

of set theory, “exist objectively and independently of our mental acts and decisions”.[14] But “they do not belong to the physical world, and even their indirect connection with physical experience is very loose”.[15] Still, “despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory”.[16] There is no “reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception, which induces us to build up physical theories”.[17] For mathematical intuition “is sufficiently clear to produce the axioms of set theory and an open series of extensions of them”.[18]

But how do we get a mathematical intuition of sets sufficiently clear to produce the axioms of set theory? According to Gödel, we may “extend our knowledge of these abstract concepts”, that is, “make these concepts themselves precise” and “gain comprehensive and secure insight into the fundamental relations that subsist among them, i.e., the axioms that hold for them”, not “by trying to give explicit definitions for concepts and proofs for axioms” – otherwise “one would have nothing from which one could define or prove” – but “rather by cultivating (deepening) knowledge of the abstract concepts themselves”.[19] The procedure must consist “in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc.”.[20] This will “produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us”.[21] Thus we will get an “intuitive grasping of ever newer axioms”.[22]

But saying that axioms are justified if they are based on mathematical intuition is inadequate. For suppose that, by Gödel’s procedure of focusing more sharply on the concepts concerned, you get an intuition of the concept of set, say Σ. Let *S* be a formal system for set theory whose axioms such an intuition ensures you to be true of Σ. Since Σ is a model of *S*, obviously *S* is consistent. Then, by Gödel’s first incompleteness theorem, there is a sentence *G* of *S* true of Σ but unprovable in *S*. Thus $S \cup \{\neg G\}$ is consistent, and hence has a model, say Σ’. Then Σ and Σ’ are both models of *S*, but *G* is true of Σ and false of Σ’, and so Σ and Σ’ are not equivalent. Now if, again by Gödel’s procedure, you focus more sharply on the way you obtained Σ’, you get

[14] Gödel 1986-, III, p. 311.
[15] *Ibid*., II, p. 267.
[16] *Ibid*., II, p. 268.
[17] *Ibid*.
[18] *Ibid*.
[19] Ibid., III, p. 383.
[20] *Ibid*.
[21] *Ibid*.
[22] *Ibid*., III, p. 385.

an intuition of the concept of set Σ’. Then you have two different intuitions, one ensuring that Σ is the genuine concept of set, and the other ensuring that Σ’ is the genuine concept of set, where the sentence *G* is true of Σ and false of Σ’. Which of Σ and Σ’ is the genuine concept of set? Gödel’s procedure provides no answer.

On the other hand, Hilbert claims that axioms are justified if their consistency can be established by means of a proof based on intuition. By that he means pure sensible intuition – what Kant, though with a different meaning, calls “intuition sensible but pure”.[23] According to Hilbert, axioms must be complete, i.e., all mathematical truths of the field concerned must be derivable “from the axioms by means of a finite number of logical inferences”.[24] Moreover, axioms must be consistent, in the sense that “a finite number of logical steps based upon them can never lead to contradictory results”.[25] The importance of consistency derives from the fact that, “if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true”.[26] Thus “‘consistent’ is identical to ‘true’”.[27]

But, according to Hilbert, “we can never be certain in advance of the consistency of our axioms if we do not have a special proof of it”.[28] To be indisputable, such a special proof must use absolutely reliable methods, and the latter are those based on pure sensible intuition, which is “a kind of intuitive insight”.[29] Thus “the most general and fundamental idea of the Kantian epistemology retains its significance: to ascertain the *a priori* intuitive mode of thought”.[30] By that, however, Hilbert means something very different from Kant’s construction of mathematical concepts. For he claims that “something is already given to us in advance in our faculty of representation: certain extra-logical concrete objects that exist intuitively as an immediate experience before all thought”.[31] These objects are “completely surveyable in all their parts, and their presentation, their differences, their succeeding one another or their being arrayed next to one another is immediately and intuitively given to us, along with the objects”.[32] Thus Hilbert’s view of pure sensible intuition is very un-Kantian.

But saying that axioms are justified if their consistency can be established by means of a proof based on pure sensible intuition, is inadequate. For, by Gödel’s second incompleteness theorem, for any

[23] Kant 1900-, II, p. 410.
[24] Hilbert 1996a, p. 1095.
[25] Hilbert 2000, p. 250.
[26] Hilbert 1980, p. 39.
[27] Hilbert 1931, p. 122.
[28] Hilbert 1996c, p. 1120.
[29] Hilbert 1996e, p. 1161.
[30] Hilbert 1996d, pp. 1149-1150.
[31] *Ibid*., p. 1150.
[32] *Ibid*.

formal system containing elementary number theory, no consistency proof based on pure sensible intuition is possible. Hilbert needn't have waited for Gödel to realize that. For Kant had already stressed that "it is, to be sure, a necessary logical condition" that a given concept "must contain no contradiction; but this is not by any means sufficient to guarantee the objective reality of the concept, that is, the possibility of such an object as is thought through the concept".[33]

Moreover, saying that axioms must be complete is inadequate. For, by Gödel's first incompleteness theorem, every consistent formal system containing elementary number theory is incomplete. Again, Hilbert needn't have waited for Gödel to realize that. For Kant had already stressed that the mathematician "arrives at an illuminating and at the same time general solution of the problem through a chain of inferences that is always guided by intuition".[34] Demonstrations, "as the term itself indicates, proceed through the intuition of the object".[35] An "apodictic proof can be called a demonstration only insofar as it is intuitive".[36] Thus, in the derivation of geometrical theorems we always need new geometrical intuitions, and therefore a purely logical derivation from a finite number of axioms is impossible. Kant's view is shared by Gödel, with a difference. According to Gödel, the assertion that "in the derivation of geometrical theorems we always need new geometrical intuitions, and that therefore a purely logical derivation from a finite number of axioms is impossible", is "incorrect if taken literally".[37] But "if in this proposition we replace the term 'geometrical' by 'mathematical' or 'set-theoretical', then it becomes a demonstrably true proposition".[38]

Furthermore, saying that 'consistent' is identical to 'true' is inadequate. For, by a corollary to Gödel's first incompleteness theorem, every consistent formal system containing elementary number theory has a consistent extension in which some falsity is provable. So consistency is not sufficient for truth. Again, Hilbert needn't have waited for Gödel to realize that. For Kant had already stressed that "a judgment, even though it is free of any internal contradiction, can still be either false or groundless".[39]

It is no escape from Gödel's incompleteness results to claim that, although we must give up "introducing the idea of a total system for mathematics", it is nonetheless possible to consider "the actually existing system of analysis and set theory as providing an adequate framework for accommodating the geometrical and physical disciplines. A formalism may correspond to this aim even without having the property of full

[33] Kant 1900-, III, p. 187.
[34] *Ibid*., III, p. 471.
[35] *Ibid*., III, p. 482.
[36] *Ibid*., III, p. 481.
[37] Gödel 1986-, III, p. 385.
[38] *Ibid*.
[39] Kant 1900-, III, 141.

deductive closure".[40] This is no escape because, even allowing that the actually existing system of analysis and set theory provides such an adequate framework, still, by Gödel's second incompleteness theorem, no consistency proof based on pure sensible intuition is possible for such a system. So the axioms of the system are unjustified.

Similarly, it is no escape from Gödel's incompleteness results to claim that, although no single consistent formal system containing elementary number theory can be complete, we may "avoid as far as possible the effects of Gödel's theorem" if we start from a given incomplete system and "obtain a more complete one by the adjunction as axioms of formulae, seen intuitively to be correct, but which the Gödel theorem shows are unprovable in the original system"; then from this we obtain "a yet more complete system by a repetition of the process, and so on".[41] This process will be continued into the transfinite, associating a system "with any constructive ordinal".[42] This is no escape because the resulting sequence of formal systems will be incomplete.

To justify the axioms of set theory, Gödel and Hilbert appeal to different kinds of intuition, and in different ways.

Gödel thinks it possible to justify the axioms of set theory directly in terms of pure intellectual intuition, since the latter is pure intuition of abstract objects, and sets are abstract objects. In Gödel's view, it is by pure intellectual intuition that "the axioms force themselves upon us as being true".[43]

On the other hand, Hilbert thinks it possible to justify the axioms of set theory indirectly in terms of pure sensible intuition – by a consistency proof – since pure sensible intuition is pure intuition of concrete objects. A justification of the axioms of set theory can only be indirect since sets are abstract objects. So no direct justification in terms of pure sensible intuition is possible. In Hilbert's view, a consistency proof based on pure sensible intuition "provides us with a justification for the introduction" of propositions concerning abstract objects such as the axioms of set theory, which Hilbert calls the "ideal propositions".[44]

Gödel's and Hilbert's appeals to intuition can be viewed in a perspective which makes the above comparison between the axiomatic method and theology less haphazard than it might appear.

For, appealing to pure intellectual intuition to justify the axioms of set theory directly, Gödel assigns pure intellectual intuition a power comparable to that Aquinas assigns to God. Aquinas claims that God has the entire knowledge of a thing "by understanding the simple essence".[45] He knows it "with the knowledge of vision", and "the present intuition of

[40] Hilbert-Bernays 1968-70, II, p. 289.
[41] Turing 1939, p. 198.
[42] *Ibid*., p. 161.
[43] Gödel 1986-, II, p. 268.
[44] Hilbert 1967, p. 471.
[45] Thomas Aquinas, *Summa Theologiae,* I, q. 85, a. 5.

God extends over all time, and to all things which exist in any time, as to objects present to him".[46] Similarly Gödel claims that, focusing more sharply on the concepts concerned, we know sets by understanding the simple essence. We know them with the knowledge of vision, and our intuition extends to all sets as to objects present to us. As we have seen, this claim is refuted by Gödel's first incompleteness theorem.

On the other hand, appealing to pure sensible intuition to justify the axioms of set theory indirectly, Hilbert assigns consistency proofs based on pure sensible intuition a role similar to that Spinoza assigns to the ideas of God. Spinoza claims that, "if an architect conceives a building properly constructed, though such a building never existed, and also will never exist, nevertheless the idea of such a building is true; and the idea remains the same, whether the building exists or not".[47] In other words, coherent ideas of the mind are true. For "our mind, in so far as it perceives things truly, is part of the infinite intellect of God".[48] So coherent "ideas of the mind are as necessarily true as the ideas of God".[49] Similarly, Hilbert claims that arbitrarily given axioms which are proved to be consistent by means of a proof based on pure sensible intuition, are true. For consistent ideas of the mind are true. As we have seen, this claim is refuted by a corollary to Gödel's first incompleteness theorem.

Since all known justifications of axioms in terms of intuition are inadequate, we may conclude that the view that the axiomatic method expresses the real nature of mathematics is incapable of providing a satisfactory answer to the question how axioms are justified.

Against such conclusion it might be objected that it rests on the unproven assumption that any justification of axioms must appeal to some kind of intuition. On the contrary, there is a justification of axioms which does not appeal to intuition: axioms are justified if true consequences follow from them. Therefore it is the consequences that give the reasons why we believe the axioms.

Such an alternative justification of axioms in terms of their consequences has been put forward by several people.

For example, Zermelo claims that "principles must be judged from the point of view of science", i.e., from the point of view of their consequences, "and not science from the point of view of principles fixed once and for all".[50] In particular, the principle of choice is justified because there is "a number of elementary and fundamental theorems and problems" that "could not be dealt with at all without the principle of choice".[51] So long as it "cannot be definitely refuted, no one has the right

[46] *Ibid*., I, q. 14, a. 9.
[47] Spinoza 1925, II, p. 26.
[48] *Ibid*., II, p. 125.
[49] *Ibid*.
[50] Zermelo 1967, p. 189.
[51] *Ibid*., p. 188.

to prevent the representatives of productive science from continuing to use this 'hypothesis'".[52]

Similarly, Gödel claims that, "even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its 'success'", where success means "fruitfulness in consequences, in particular in 'verifiable' consequences".[53] Thus, "besides mathematical intuition, there exists another (though only probable) criterion of the truth of mathematical axioms, namely their fruitfulness in mathematics and, one may add, possibly also in physics".[54]

But a justification of axioms in terms of their consequences is inadequate, because the fact that true consequences follow from the axioms provides no justification for them: true consequences can follow from false axioms.

As Kant states, "inferring the truth of a cognition from the truth of its consequences would be admissible only if all its possible consequences are true", but "this is an unfeasible procedure, since to discern all possible consequences of any accepted proposition exceeds our powers".[55]

Evidence for Kant's statement is provided, for example, by the fact that the set of all consequences of the axioms of second-order Peano arithmetic PA^2 is not algorithmically enumerable. For, since the axioms of PA^2 are categorical, a second-order sentence is a consequence of the axioms of PA^2 if and only if it is true of the natural numbers. Thus, if the set of all consequences of the axioms of PA^2 were algorithmically enumerable, so would be the set of all second-order sentences true of the natural numbers. But, by Tarski's theorem for second-order sentences, the set of all second-order sentences true of the natural numbers is not definable in the set of all natural numbers by any second-order formula, and so *a fortiori* it is not algorithmically enumerable. Then the set of all consequences of the axioms of PA^2 is not algorithmically enumerable. Thus there exists no algorithmic procedure, *a fortiori* no feasible procedure, for enumerating all consequences of the axioms of PA^2. Therefore, as Kant states, inferring the truth of a cognition from the truth of its consequences is an unfeasible procedure.

Moreover, a justification of the axioms in terms of their consequences does not fit in with the view that mathematics consists in the deduction of conclusions from given axioms. For such a view involves that, since axioms are what their consequences depend on, it is the axioms that give the reasons for believing the consequences. In particular, according to Hilbert, it is the consistency of the axioms that

[52] *Ibid*., p. 189.
[53] Gödel 1986-, II. p. 261.
[54] *Ibid*., II, p. 269.
[55] Kant 1900-, III, p. 514.

gives the reason for believing the consequences. Therefore, when supporters of the view that mathematics consists in the deduction of conclusions from given axioms claim that axioms can be justified in terms of their consequences, they speak rather incoherently.

To claim that it is the consequences that give the reasons for believing the axioms one must drop the view that mathematics consists in the deduction of conclusions from given axioms. Actually, one would be well advised to drop that view anyway, because it "does not correspond to simple observation. If the Pythagorean theorem were found to not follow from postulates, we would again search for a way to alter the postulates until it was true. Euclid's postulates came from the Pythagorean theorem, not the other way".[56]

Since a justification of the axioms in terms of their consequences is inadequate, we may finally conclude that the view that mathematics consists in the deduction of conclusions from given axioms is incapable of providing a satisfactory answer to the question how axioms are justified.

On the other hand, such a view is also incapable of providing a satisfactory answer to the question how axioms are discovered. For, as Aristotle points out, it is "impossible to say anything" about the principles of each science "on the basis of the proper starting points of the science in question, since the starting points are primary to everything".[57] To explain how principles are discovered one needs a method that, "being fit for inquiring, possesses the path to the principles of all disciplines".[58] Such a method is not provided by the axiomatic method. As we will argue in the next section, it is provided by the analytic method.

3. Analytic Method and Discourse

Since antiquity, the main reason for an axiomatic presentation of mathematics has been didactic, i.e., its efficiency in transmitting knowledge in the form of textbooks. But mathematics cannot be reduced to its didactic presentation. Therefore, the assertion that mathematics consists in the deduction of conclusions from given axioms, rather than expressing the real nature of mathematics, at most expresses the real nature of teaching.

This has been stressed by several people. For example, Zermelo distinguishes mathematics from its didactic presentation arguing that "geometry existed before Euclid's *Elements*, just as arithmetic and set theory did before Peano's *Formulaire*, and both of them will no doubt

[56] Hamming 1980, p. 87.
[57] Aristotle, *Topica*, A 2, 101 a 36-b 1.
[58] Ibid., A 2, 101 b 3-4.

survive all further attempts to systematize them in such a textbook manner".[59]

Actually, teaching is the end of mathematical research – the end not in the sense that it is the goal towards which the activity of progressing through the steps in this sequence is geared, but only in the sense that it is the last term in a sequence. Therefore, it does not express the real nature of mathematics.[60] To express it we must look elsewhere.

Specifically, we must look to the analytic method, a rival of the axiomatic method originally used by Hippocrates of Chios to solve certain mathematical problems, such as cube duplication or the quadrature of certain lunes, and explicitly described by Plato, who used it to solve certain philosophical problems, such as the question whether virtue is teachable or whether the soul is immortal.

According to Plato, to solve a problem, "on each occasion I assume the hypothesis which I judge to be the strongest, and I lay down as true whatever seems to me to agree with it", while "I put down as not true whatever does not seem to me to agree with it".[61] However, once you had assumed a hypothesis, you wouldn't go on until "you had investigated its consequences, to see whether they agreed or disagreed with one another".[62] Moreover, you would have to give an account of the hypothesis itself. Now, "to give an account of the hypothesis, you would give it in the same way, assuming another hypothesis, whichever among higher hypotheses seemed best, until you came to something" provisionally "sufficient".[63] And so on, ad infinitum.

However, Plato's description of the analytic method, provides no indication as to how hypotheses are obtained. A more complete description can be given as follows.

To solve a mathematical problem, we formulates a hypothesis that is a condition sufficient for its solution. The hypothesis is obtained from the problem, and possibly other data, by non-deductive inferences – inductive, analogical, etc..[64] However, the hypothesis must not only be a condition sufficient for the solution of the problem but must also be plausible. That is, it must be compatible with the existing knowledge, in the sense that, comparing the reasons for and the reasons against the hypothesis on the basis of the existing knowledge, the reasons for the hypothesis prevail over those against it. But the hypothesis is in turn a problem that must be solved, and will be solved much in the same way, i.e. formulating another hypothesis that is a condition sufficient for its solution, and is obtained from the previous hypothesis, and possibly

[59] Zermelo 1967, p. 189.

[60] For further discussion of the didactic character of axiomatic presentations, see Cellucci 1998, pp. 127-134.

[61] Plato, *Phaedo*, 100 a 3-7.

[62] *Ibid.*, 101 d 4-5.

[63] *Ibid*., 101 d 5-e 1.

[64] On different kinds of non-deductive inferences by which hypotheses can be obtained, see Cellucci 2002, Part IV.

other data, by non-deductive inferences. And so on, ad infinitum. Therefore, the solution of a mathematical problem is an essentially infinite process.

Such a description of the analytic method must be supplemented by a number of remarks.

1. The fact that assessing the plausibility of a hypothesis involves comparing the reasons for and the reasons against the hypothesis on the basis of the existing knowledge, does not mean that the existing knowledge is final. For, in the process of comparing the reasons for and the reasons against the hypothesis, certain long-standing hypotheses, on which the existing knowledge depends, may turn out to be no longer plausible. That depends on the fact that the investigation concerning the plausibility of the new hypothesis may bring such new data to light, or open such new perspectives, that the balance between the reasons for and the reasons against the long-standing hypotheses is reversed: the reasons against end up prevailing over those for them. In that case, it becomes necessary to modify the long-standing hypotheses or even to drop them.

Of course, when confronted with the choice of adopting a new hypothesis or dropping long-standing ones, there is a natural tendency to stick to the latter. But, in the presence of overwhelming evidence, even long-standing hypotheses will in the end be modified, or even dropped.

As Kant points out, a person may be reluctant to "get rid of false hypotheses because he has contrived them and they seem so probable to him", just "like a man who has brought a child up with much effort and care and who afterwards does not want it to go away, so as not to lose all his work, effort, and expense".[65] In fact, when a false consequence is found, one "need not let his spirits sink, just as the alchemist always keeps working on the hypothesis of making gold".[66] Admittedly, "if a cognition has a single false consequence, then it is totally false, even though some right consequences can be derived from it".[67] But even then there is no reason to despair. False hypotheses "serve to get true hypotheses fabricated in subsequent ages, for one who is familiar with all possible false paths cannot possibly fail to find the right path at last".[68] Therefore one must try to make the most of false hypotheses.[69]

2. Since comparing the reasons for and the reasons against a hypothesis generally involves considering the consequences of the hypothesis, in the analytic method it is the consequences that give the reasons for believing the hypotheses rather than the other way round.

[65] Kant 1900-, XXIV, p. 224.
[66] *Ibid*., XXIV, p. 889.
[67] Kant 1998, I, p. 87.
[68] Kant 1900-, XXIV, p. 225.
[69] On Kant's views concerning hypotheses, probability and verisimilitude, see Capozzi 2002, Ch. 15.

Thus, while justifying axioms in terms of their consequences does not fit in with the axiomatic method, it fits in with the analytic method.

This is pointed out by Russell, who argues that "we tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true".[70] Now, "the inferring of premises from consequences is the essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science".[71] The "usual mathematical method of laying down certain premises and proceeding to deduce their consequences, though it is the right method of exposition", i.e., the right didactic method, "does not, except in the more advanced portions, give the order of knowledge".[72] The actual order of knowledge is, first, "the registration of 'facts'", then "the inductive discovery of hypotheses, or logical premises, to fit the facts", and finally "the deduction of new propositions from the facts and hypotheses".[73] Thus there is a "close analogy between the methods of pure mathematics and the methods of the sciences of observation".[74]

Russell's version of the analytic method has a definite Aristotelian flavor. For Aristotle claims that "induction is the starting-point which knowledge even of the universal presupposes", since "there are starting-points from which syllogism proceeds, which are not reached by syllogism; it is therefore by induction that they are acquired".[75] Such a version of the analytic method, however, is an essentially incomplete one. For, in the analytic method, hypotheses may be obtained not only by induction but also by other kinds of non-deductive inferences. Moreover, the justification of hypotheses may proceed not only downwards, considering their consequences, but also upwards, formulating new hypotheses.

3. That, in the analytic method, the solution of mathematical problems is an essentially infinite process, does not mean that the process of passing from one hypothesis to another cannot stop temporarily. In fact, it stops temporarily at each step. For, to assess the hypothesis formulated at a given step, one must compare the reasons for and the reasons against it, and that may require a long time, since it may involve considering many consequences of the hypothesis. In any case, the process will always stop only temporarily. Sooner or later it will have to start again, since every hypothesis is a problem, and a problem that must be solved. The fact that, at a given step, the reasons for a hypothesis prevail over those against it, provides only a temporary support for the hypothesis that can be reversed at any time. For their prevailing is only

[70] Russell 1973, pp. 273-274.
[71] *Ibid*., p. 274.
[72] *Ibid*., p. 282.
[73] *Ibid*.
[74] *Ibid*., p. 272.
[75] Aristotle, *Ethica Nicomachea*, VI 3, 1139 b 28-31.

relative to the existing knowledge, and may be upset if new data are brought to light or new perspectives are opened.

4. The analytic method is both a method of discovery and a method of justification. This depends on the fact that a peculiar character of non-deductive inferences is that they allow to infer different conclusions from the very same premisses. For example, from the premiss, ‘All the digits in the decimal expansion of π computed so far are random’, one may inductively infer both ‘All the digits in the decimal expansion of π are random’ and ‘All the digits in the decimal expansion of π computed so far are random, but those which will computed in the future will consist of all 9’s’. Since non-deductive inferences allow to infer different conclusions from the very same premisses, to find a suitable hypothesis one must choose between different conclusions. That requires a careful assessment of the reasons for and the reasons against each conclusion. Such an assessment is a process of justification, therefore justification is part of discovery. This blurs the distinction between discovery and justification and makes the analytic method both a method of discovery and a method of justification. Thus the distinction between discovery and justification loses its theoretical importance.

Supporters of the view that the axiomatic method expresses the real nature of mathematics motivate such distinction arguing that discovery escapes logical analysis. According to them, one must distinguish sharply between the process of discovering new mathematical results and the process of justifying mathematical results already discovered. The former is purely subjective and is left to psychological analysis, the latter is more definite and may be subject to logical analysis.

For example, Frege claims that “it not uncommonly happens that we first discover the content of a proposition, and only later give the rigorous proof of it”, therefore in general “the question of how we arrive at the content of a judgement should be kept distinct from the other question, Whence do we derive the justification for its assertion?”.[76] The “first question may have to be answered differently for different persons; the second is more definite, and the answer to it is connected with the inner nature of the proposition considered”.[77] Therefore the first question is purely psychological, only the second one “is removed from the sphere of psychology”.[78]

This, however, overlooks that, if discovery escapes logical analysis, then justification too escapes it. For, as we have seen, the view that the axiomatic method expresses the real nature of mathematics is incapable of providing a logical analysis of the justification of the axioms, and, without such a justification, the axiomatic method does not justify anything.

[76] Frege 1959, p. 3.
[77] Frege 1967, p. 5.
[78] Frege 1959, p. 3.

5. While intuition plays an essential role in the axiomatic method, it plays no role in the analytic method, either in the discovery or in the justification of hypotheses. For a hypothesis for the solution of a given problem is obtained from the problem, and possibly other data, by means of non-deductive inferences, thus not by intuition but by discourse. Moreover, the plausibility of the hypothesis is established comparing the reasons for and the reasons against it, thus not by intuition but by discourse. Therefore, in the analytic method, intuition is replaced by discourse.

This entails that the analytic method cannot be absolutely reliable. For, the conclusions of the non-deductive inferences by which hypotheses are obtained do not follow from their premisses with absolute necessity, since non-deductive inferences are ampliative. Moreover, the process by which the plausibility of hypotheses is established is not absolutely reliable. For it consists in comparing the reasons for and the reasons against them, and such a comparison depends on the knowledge existing at that moment, which is not absolutely reliable. Therefore mathematics cannot be absolutely certain.

Supporters of the view that the axiomatic method expresses the real nature of mathematics claim that, being based on intuition, the axiomatic method is absolutely reliable. For, in their opinion, intuition provides an absolutely reliable justification for the axioms, and conclusions of deductive inferences follow from their premisses with absolute necessity. Therefore mathematics is absolutely certain. But this claim is unfounded because, as we have seen, all known justifications of axioms in terms of intuition are inadequate.

6. The analytic method should not be confused with the analytic-synthetic method, originally stated by Aristotle and restated by Pappus and others in terms of a different view of the direction of analysis.[79] Such two methods are essentially different.[80] As Lakatos points out, in the analytic method, analysis is a means of finding unknown hypotheses, and proceeds "without any known lemmas, without any safe axiomatic systems".[81] On the contrary, in the analytic-synthetic method, analysis is simply "a heuristic pattern in already axiomatized Euclidean geometry".[82] As a means of finding hypotheses it loses "its function; when used at all", it is "only a heuristic device for mobilizing the – already proven or trivially valid – lemmas necessary for the synthesis".[83] Analysis is "not any more a venture into the unknown", but only "an exercise in mobilizing and ingeniously connecting the

[79] On Aristotle's form of the analytic-synthetic method, see Byrne 1997. On Pappus' variant, see Hintikka-Remes 1974, Knorr 1993.
[80] This is usually overlooked; e.g., see Mueller 1992, Menn 2002. On the distinction between these two methods, see Cellucci 1998, Ch. 8.
[81] Lakatos 1978, II, p. 99.
[82] *Ibid*., II, p. 100.
[83] *Ibid*.

relevant parts of the known. The lemmas which were once daring and often falsified conjectures harden into auxiliary theorems".[84]

While the analytic-synthetic method, being a heuristic pattern in already axiomatized Euclidean geometry which can be used only for finding proofs of given propositions from given axioms, essentially depends on the axiomatic method, the analytic method is independent of it and indeed alternative to it. Actually, the axiomatic method is what results from the analytic method if, at a certain stage, the process of passing from one hypothesis to another is stopped definitively, the hypothesis reached at that step being considered no longer as a problem to be solved but as an absolutely unproblematic starting point. Therefore the axiomatic method is an unjustified truncation of the analytic method which removes the most important part of it.

This explains Plato's vehement attack against the axiomatic method. He claims that mathematicians practising the axiomatic method use hypotheses improperly, because they "take them for granted, as axiomatic principles, and do not think it necessary to give any account of them either to themselves or to others, considering them as absolutely evident. Then, starting from these hypotheses and developing their consequences, they arrive at last at the result they were aiming at, and conclude 'What it was required to do'".[85] Since they do not give any account of the hypotheses, they "only dream about being, and never can they behold the waking reality so long as they leave the hypotheses which they use unexamined, and are unable to give an account of them".[86] But, "when a man does not know his own starting-point, and when the conclusion and intermediate steps are also woven together out of unknown material, how can he imagine that such a fabric of convention can ever become science?"[87]

7. It is interesting to note that supporters of the view that the axiomatic method expresses the real nature of mathematics often make claims that don't fit in with the axiomatic method but only with the analytic method.

An example is provided by Hintikka's claim that proving a theorem from given second-order axioms might involve the discovery of new valid rules of inference, which in turn might involve the discovery of new axioms. Another example is provided by Zermelo's claim that the axiom of choice is justified by its consequences.

An even more significant example is provided by Hilbert's claim that, in solving mathematical problems, we "do not habitually follow the chain of reasoning back to the axioms in arithmetical discussions, any more than in geometrical".[88] Rather, we proceed "by

[84] *Ibid*.
[85] Plato, *Respublica*, VI 510 c 6-d 3.
[86] *Ibid*., VII 533 b 8-c 3.
[87] *Ibid*., VII 533 c 3-5.
[88] Hilbert 2000, p. 246.

means of a finite number of steps based upon a finite number of hypotheses which are implied in the statement of the problem and which must be exactly formulated".[89] Such hypotheses are temporarily taken "as the axioms of the individual fields of knowledge".[90] But the solution thus obtained is "only temporary. In fact, in the individual fields of knowledge" the need soon arises "to ground the fundamental axiomatic propositions themselves".[91] Thus one gives 'proofs' of them. However, "critical examination of these 'proofs' shows that they are not in themselves proofs, but basically only make it possible to trace things back to certain deeper propositions, which in turn are now to be regarded as new axioms instead of the propositions to be proved".[92] And so on. Therefore, we solve mathematical problems by the analytic method.

Surprisingly enough, Hilbert even states that the "regressive method", i.e., the analytic method, "finds its perfect expression in what is called today the 'axiomatic method'".[93] On the other hand, however, he states: "We call the development of a theory axiomatic, when its basic concepts and basic assumptions are put as such at its beginning, and the remaining content of the theory is derived logically from them by means of definitions and proofs. In this sense, geometry was axiomatically founded by Euclid".[94]

Such patently inconsistent statements can be reconciled only if one takes into account that they were made at different stages. Originally, as a working mathematician, Hilbert was naturally led to think that we solve mathematical problems by the analytic method. But his attitude drastically changed following Weyl's and Brouwer's attacks against classical mathematics, which threatened "to dismember and mutilate our science".[95] Then his main worry became "to regain for mathematics the old reputation for incontestable truth".[96] He thought that the method to achieve that aim was "none other than the axiomatic".[97] Then 'axiomatic method' could no longer designate the analytic method. Unfortunately for Hilbert, however, Gödel's incompleteness results showed that the axiomatic method was unequal to Hilbert's intended aim.

4. Demonstrative and Non-Demonstrative Reasoning

The contrast between the axiomatic method and the analytic method is in particular a contrast between the two kinds of reasoning on which the two methods are based, i.e., demonstrative and non-demonstrative

[89] *Ibid*. p. 244.
[90] Hilbert 1996b, p. 1108.
[91] *Ibid*., p. 1109.
[92] *Ibid*.
[93] Hilbert 1992, p. 18.
[94] Hilbert-Bernays 1968-70, I, p. 1.
[95] Hilbert 1996c, p. 1119.
[96] *Ibid*.
[97] *Ibid*.

reasoning, respectively. Demonstrative reasoning is the deductive derivation of conclusions from premisses which are primitive and true, in some sense of ‘true’. Non-demonstrative reasoning is the non-deductive derivation of conclusions from premisses which are not known to be true but are only accepted opinions, i.e., plausible propositions.

That non-demonstrative reasoning is the non-deductive derivation of conclusions from premisses which are not known to be true originates the objection against the analytic method that, while demonstrative reasoning, on which the axiomatic method is based, is cogent, non-demonstrative reasoning, on which the analytic method is based, is not cogent.

Building on such objection an influential tradition, from Aristotle to Polya, claims that there exists a sharp distinction between demonstrative and non-demonstrative reasoning, and that the former is essentially superior to the latter.

According to such tradition, “there are two kinds of reasoning”, i.e., “demonstrative reasoning” and non-demonstrative reasoning – what Aristotle calls ‘dialectical’ and Polya “plausible reasoning”.[98] Demonstrative reasoning “starts from premisses that are true and primitive”.[99] It draws conclusions from them by syllogism, which is truth-preserving. Therefore demonstrative reasoning is “safe, beyond controversy, and final”.[100] Conclusions obtained by means of it are absolutely certain. For that reason, “we secure our mathematical knowledge by demonstrative reasoning”.[101] On the other hand, non-demonstrative reasoning starts “from opinions that are accepted”, i.e., “shared by everyone, or by most people, or by the wise, and of them, by all, or by most, or by the best known and illustrious”.[102] It draws conclusions from them not only by syllogism but also by induction, for “there is on the one hand induction, on the other syllogism”.[103] Therefore non-demonstrative reasoning “is hazardous, controversial, and provisional”.[104] Conclusions obtained by means of it are not absolutely certain. Admittedly, demonstrative reasoning is “incapable of yielding essentially new knowledge about the world around us”.[105] And “anything new that we learn about the world involves plausible reasoning”.[106] I.e., it involves non-demonstrative reasoning. In particular, the latter provides “the path to the principles of all sciences”.[107] But “the result of the

[98] Polya 1954, I, p. vi.
[99] Aristotle, *Topica*, A 1, 100 a 27.
[100] Polya 1954, I, p. v.
[101] *Ibid*.
[102] Aristotle, *Topica*, A 1, 100 a 30, 100 b 21-23.
[103] *Ibid*., A 12, 105 a 11-12.
[104] Polya 1954, I, p. v.
[105] *Ibid*.
[106] *Ibid*.
[107] Aristotle, *Topica*, A 2, 101 b 3-4.

mathematician's creative work is demonstrative reasoning", which "is his profession and the distinctive mark of his science".[108]

The Aristotle-Polya tradition is still so influential that, consciously or unconsciously, most contemporary debate concerning the relations between demonstrative and non-demonstrative reasoning derives from it. However, its claim that there exists a sharp distinction between demonstrative and non-demonstrative reasoning, and that the former is essentially superior to the latter, is untenable. This can be shown as follows.

To know whether an argument is demonstrative, one must know whether its premisses are true. But knowing whether they are true is generally impossible. This is implicit in what we have already said, but there is no harm in repetition.

That the premisses are true can be meant either in Gödel's strong sense that they express properties of objects independent of us, or in Hilbert's weak sense that they are consistent.

If the premisses are true in Gödel's strong sense that they express properties of objects independent of us, then they have a model, which consists of such objects. However, by Gödel's first incompleteness theorem, the proposition, 'The premisses have a model', is not provable from them but only from a proper extension of them. If the premisses of such proper extension are true in Gödel's strong sense, then they have a model. However, by Gödel's first incompleteness theorem, the proposition, 'The premisses of the proper extension have a model', is not provable from them but only from a proper extension of them. And so on, ad infinitum. Therefore, it is impossible to know whether the premisses are true in Gödel's strong sense.

On the other hand, if the premisses are true in Hilbert's weak sense that they are consistent, then, by Gödel's second incompleteness theorem, the consistency of the premisses is not provable from them but only from a proper extension of them. If the premisses of such proper extension are true in Hilbert's weak sense, then, by Gödel's second incompleteness theorem, the consistency of the premisses of the proper extension is not provable from them but only from a proper extension of them. And so on, ad infinitum. Therefore, it is impossible to know whether the premisses are true in Hilbert's weak sense.

We may then conclude that knowing whether the premisses are true is generally impossible.

This is often overlooked. For example, Dummett claims that, for deductive reasoning "to be fruitful, we must be able to grasp the premisses and acknowledge them as true without perceiving the possibility of drawing" the "conclusion".[109] He also claims that deductive reasoning is "astonishingly fruitful".[110] Therefore, Dummett implicitly

[108] Polya 1954, I, p. vi.
[109] Dummett 1991, p. 305.
[110] *Ibid.*, p. 306.

assumes that we are able to grasp the premisses and acknowledge them as true.

Since knowing whether the premisses are true is generally impossible, the premisses of demonstrative arguments are only accepted opinions. So they have the same status as the premisses of non-demonstrative arguments. Thus our reliance on demonstrative reasoning ultimately rests on non-demonstrative reasoning, if not on faith.

This is acknowledged even by Polya, who admits that non-demonstrative reasoning, and specifically "analogy and particular cases", perhaps "not only help to shape the demonstrative argument and to render it more understandable, but also add to our confidence in it. And so we are led to suspect that a good part of our reliance on demonstrative reasoning may come from plausible reasoning".[111]

That it is impossible to know whether the premisses are true disposes of the objection against the analytic method that, while demonstrative reasoning is cogent, non-demonstrative reasoning is not cogent. Such an objection is untenable, because demonstrative reasoning cannot be more cogent than the premisses from which it starts. But the premisses cannot be cogent, since knowing whether they are true is generally impossible. So they are only accepted opinions, and therefore have the same status as the premisses of non-demonstrative reasoning. This blurs the distinction between demonstrative and non-demonstrative reasoning.

Moreover, not only it is impossible to know whether the premisses are true, but it is also impossible to justify deductive inferences in any absolute sense. This disposes of the further objection against the analytic method that, while deductive inferences can be justified, non-deductive inferences cannot be justified. Such an objection is untenable, because deductive inferences can be justified as much, or as little, as non-deductive inferences. Indeed, they can be justified much in the same sense as non-deductive inferences, specifically, in a sense that is by no means absolute.[112] In fact, the question of justifying inferences is badly formulated by supporters of the view that the axiomatic method expresses the real nature of mathematics. For they assume that inferences can be justified merely referring to the internal logical structure of the inferences, whereas a justification – though a not absolute one – can be given only in terms of the role inferences play in knowledge. Again, the fact that deductive inferences can be justified much in the same sense as non-deductive inferences, blurs the distinction between demonstrative and non-demonstrative reasoning.

We may then conclude that the claim of the Aristotle-Polya tradition that there exists a sharp distinction between demonstrative and non-demonstrative reasoning, and that the former is essentially superior

[111] Polya 1954, II, p. 168.
[112] Space prevents me from discussing this matter here. I will discuss it in Cellucci 200?.

to the latter, is untenable. It must then be replaced by the claim that there is no sharp difference between demonstrative and non-demonstrative reasoning, and that the former is by no means essentially superior to the latter.

Such a claim has been actually made by another tradition, from Plato to Ramus. According to such tradition, the analytic method based on non-demonstrative reasoning, which such tradition calls 'dialectic', "is the only way which, doing away with hypotheses, is capable of taking to the starting-point itself in order to make the conclusions secure".[113] In other words, it is capable of taking us "to the vivid original forms, the archetypes".[114]

There is, however, a feature of the Plato-Ramus tradition that is definitely untenable. According to such tradition, non-demonstrative reasoning is absolutely reliable. For dialectic "shows us how not to lose the right way in arguing".[115] Armed with dialectic, one "can fight all the objections one by one and refute them, not by appeals to opinion, but to absolute truth, never faltering at any step of the argument".[116] One can do that because dialectic is capable of taking to the starting-point itself by means of intellectual intuition. Of all faculties in the soul, "intellectual intuition answers to the highest".[117] Indeed, it is able "to behold those higher things, which can only be seen with the mind's eye".[118] Therefore, conclusions obtained by means of non-demonstrative reasoning are absolutely certain.

But non-demonstrative reasoning cannot be absolutely reliable, not only because it consists of non-deductive inferences, which, being ampliative, are not absolutely reliable, but also because the process by which the plausibility of its premisses is established is not absolutely reliable. Therefore conclusions obtained by means of non-demonstrative reasoning cannot be absolutely certain.

The claim of the Plato-Ramus tradition that they are absolutely certain depends on the assumption that dialectic can "reach the starting-point of everything, which depends on no hypothesis".[119] It can "ascend to the vision of the infinite mind".[120] But such an assumption is untenable because, as we have seen, contrary to what the Plato-Ramus tradition maintains, we cannot ascend to the vision of the infinite mind by Gödel's mathematical intuition, i.e., by intellectual intuition.

That conclusions obtained by non-demonstrative reasoning cannot be absolutely certain does not contradict the above statement that demonstrative reasoning is by no means essentially superior to non-

[113] Plato, *Respublica*, VII, 533 c 7-d 1.
[114] Ramus 1964, p. 43 v.
[115] *Ibid*., p. 8 r.
[116] Plato, *Respublica*, VII, 534 c 1-3.
[117] *Ibid*., VI, 511 d 8.
[118] *Ibid*., VI, 511 a 1.
[119] *Ibid*., VI, 511 b 6-7.
[120] Ramus 1964, p. 43 v.

demonstrative reasoning. For the claim of the Aristotle-Polya tradition, that conclusions obtained by means of demonstrative reasoning are absolutely certain, is untenable. Such conclusions cannot be absolutely certain, not only because it is generally impossible to know whether the premisses of demonstrative arguments are true and deductive inference cannot be justified in any absolute sense, but also because it is generally impossible to verify the correctness of demonstrative arguments, even when they are obtained by formal inference rules. For some demonstrative arguments are so long and complex that one can never be sure that they contain no mistakes.

Thus, while the Plato-Ramus tradition mistakenly claims that conclusions obtained by means of non-demonstrative reasoning are absolutely certain, the Aristotle-Polya tradition mistakenly claims that conclusions obtained by means of demonstrative reasoning are absolutely certain.

That conclusions obtained by means of demonstrative or non-demonstrative reasoning cannot be absolutely certain, entails that mathematical knowledge cannot be absolutely certain.

Hume claims that no mathematician places "entire confidence in any truth immediately upon his discovery of it", and although his confidence increases "every time he runs over his proofs" but "still more by the approbation of his friends" and is raised "to its utmost perfection by the universal assent and applauses of the learned world", still "this gradual encrease of assurance is nothing but the addition of new probabilities".[121] Therefore, all mathematical knowledge "resolves itself into probability, and becomes at last of the same nature with that evidence, which we employ in common life".[122]

Hume's claim is a perfectly sensible one, with two changes. First, the approbation of the mathematician's friends and the universal assent and applauses of the learned world – Hume's version of Aristotle's accepted opinions – should be understood as ultimately based on an assessment of the reasons for and the reasons against the premisses from which proofs start. For the correctness of proofs essentially depends on the plausibility of such premisses. Second, 'probability' should be replaced by 'plausibility'. For probability is a quantitative thing, whereas plausibility is not. With these two changes, Hume's conclusion is converted into the conclusion that all mathematical knowledge resolves itself into plausibility, and becomes at last of the same nature with that evidence, which we employ in common life.

Against the conclusion that mathematical knowledge cannot be absolutely certain and resolves itself into plausibility it might be objected that it essentially depends on Gödel's incompleteness results. But, if mathematical knowledge cannot be absolutely certain, then Gödel's incompleteness results too are not absolutely certain. Therefore the

[121] Hume 1978, p. 180.
[122] *Ibid*., p. 181.

conclusion that mathematical knowledge cannot be absolutely certain is not absolutely certain.

Such an objection, however, neglects that using Gödel's incompleteness results to conclude that mathematical knowledge cannot be absolutely certain consists in a *reductio ad absurdum*. Let us suppose for argument's sake that mathematical knowledge is absolutely certain. Then Gödel's incompleteness results are absolutely certain. But they entail that mathematical knowledge cannot be absolutely certain. Contradiction. Therefore mathematical knowledge cannot be absolutely certain. This argument uses *reductio ad absurdum,* but the certainty of *reductio ad absurdum* is part of the assumption that mathematical knowledge is absolutely certain, for many mathematical arguments essentially depend on *reductio ad absurdum*.

That mathematical knowledge cannot be absolutely certain is inescapable because, "about things that are not perceptible", the "gods alone have a certain knowledge, humans may only form hypotheses".[123] The most they can do is to "adopt the best and least refutable of human hypotheses and, embarking on it as a kind of raft, run the risk of sailing the seas of life".[124] Any serious reflection on the nature of mathematics must start from here.[125]

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[123] Diels 1964, 24 B 1 (Alcmaeon).

[124] Plato, *Phaedo*, 85 c 8-d 2.

[125] The views expressed in this paper are strictly related to those of Cellucci 1998, 2000, 2002.

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