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Candidate Market Models and the Calibrated CIR++ Stochastic Intensity Model for Credit Default Swap Options and Callable Floaters

Damiano Brigo
Head of Credit Models
Banca IMI , SanPaolo IMI Group
Corso Matteotti 6, 20121 Milano, Italy
http://www.damianobrigo.it

This paper: http://www.damianobrigo.it/cdsmm.pdf

Talk Outline 1

- Building Blocks: Defaultable Zero Coupon Bonds;
- Fundamental Credit Derivatives: Credit Default Swap
- CDS market payoffs and risk neutral valuation;
- Reduced Form (intensity) Models;
- Modeling Tools: Time homogeneous Poisson Processes;
- Modeling Tools: Time inhomogeneous Poisson Processes (deterministic intensity models, deterministic credit spread);
- Modeling Tools: Stochastic intensity Poisson Processes (Cox processes, Stochastic Intensity Models, credit spread volatility).
- Two Important Technicalities for Pricing with reduced form models;
- CDS forward rates and analogies with LIBOR vs SWAP rates;
- CDS Market implied deterministic intensity (or credit spread);
- Fundamental Credit Products: Defaultable floaters payoff and approximated payoffs;
- Equivalence between Defaultable floaters and Postponed CDS's;
- CDS options and Callable defaultable floaters;

Talk Outline 2

- A clever numeraire and the CDS options market model (embedded stochastic intensity);
- Analogies with the Swap Market Model;
- First examples of implied CDS rates volatilities;
- Explicit stochastic intensity modeling: The SSRD Model;
- SSRD Analytic and Automatic calibration to CDS market data and interest rate data;
- Separability of the calibration procedure;
- CDS options with the SSRD model (with CIR++ stochastic intensity)
- Relationship between CIR++ parameters and implied CDS volatilites;
- Conclusions and Further Research;
- References for Further Reading (papers, internet, books)

Building Blocks for credit products: Defaultable Zero-Coupon Bonds

Similarly to the zero coupon bond $P(t,T)$ (valuation time t, maturity T) being one of the possible fundamental quantities for describing the interest-rate curve, we now consider a defaultable bond $\bar{P}(t,T)$ as a possible fundamental variable for the defaultable market.

When considering default, we have a random time τ representing the time at which a given company defaults. The value of a bond issued by the company and promising the payment of 1 unit of currency at time T, as seen from time t, is

$$\mathbf{1}_{\{\tau>t\}}\bar{P}(t,T) := \mathbb{E}\{D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\}$$

where $\mathcal{G}_t$ represents the flow of information on whether default occurred before t and if so at what time exactly, and on the default free market variables up to t.

The "indicator" function $\mathbf{1}_{\text{condition}}$ is 1 if "condition" is satisfied and 0 otherwise. In particular, $\mathbf{1}_{\{\tau>T\}}$ reads 1 if default τ did not occur before T, and 0 in the other case.

We understand then that (ignoring recovery) $\mathbf{1}_{\{\tau>T\}}$ is the correct payoff for a corporate zero-coupon bond at time T: the contract pays 1 if the company has not defaulted, and 0 if it defaulted before T.

Building Blocks for credit products: Defaultable Zero-Coupon Bonds

From zero-coupon bonds one typically builds rates.

A "defaultable forward LIBOR rate" $\bar{F}$ can be defined by analogy with the default-free forward LIBOR rate $F(t; S, T) = (1/\alpha)(P(t, S)/P(t, T) - 1)$, where α is the year fraction between S and T. Set $\bar{F}(t; S, T) := (1/\alpha)(\bar{P}(t, S)/\bar{P}(t, T) - 1)$ on $\tau > t$.

F is typically obtained as the *fair rate* at time t of a Forward Rate Agreement contract (FRA). Can we see $\bar{F}$ as the fair rate for a sort of defaultable FRA? We will see later on that we may need to consider a different quantity for this to be the case.

So in a sense the two methods, the "fair rate in a forward contract" method and the "analogous expression in P's" method can produce different defaultable rates.

To introduce the appropriate forward contract leading to a definition of "fair defaultable forward rate", we consider now the definition of (forward) credit default swap.

Fundamental Credit Derivatives: Credit Default Swaps

"It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts."
Sherlock Holmes, *A Scandal in Bohemia*, quoted by KMV's J.R. Bohn on a credit risk survey paper.

Credit Default Swaps are basic protection contracts that became quite liquid in the last few years. CDS's are now actively traded and have become a sort of basic product of the credit derivatives area, analogously to interest-rate swaps and FRA's being basic products in the interest-rate derivatives world.

As a consequence, the need is no longer to have a model to be used to value CDS's, but rather to consider a model that can be *calibrated* to CDS's, i.e. to take CDS's as inputs, in order to price more complex credit derivatives.

CDS options are not liquid yet, but the interest for these products is growing in the market. We may expect models will have to incorporate CDS options rather than price them in a near future, similarly to what happened to CDS themselves.

Fundamental Credit Derivatives: CDS's

A CDS contract ensures protection against default. Two companies "A" (Protection buyer) and "B" (Protection seller) agree on the following.

If a third company "C" (Reference Credit) defaults at time τ, with $T_a < \tau < T_b$, "B" pays to "A" a certain (deterministic) cash amount Z. In exchange for this, "A" pays to "B" a rate R at times $T_{a+1}, \ldots, T_b$ or until default. Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.

Protection	"B"	$\rightarrow$	protection Z at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$	"A"	Prot.
Seller	"B"	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\leftarrow$	"A"	Buyer

(protection leg and premium leg respectively). The cash amount Z is a *protection* for "A" in case "C" defaults. Typically $Z =$ notional amount, or "$Z =$ notional $-$ recovery".
A typical stylized case occurs when "A" has bought a corporate bond issued by "C" and is waiting for the coupons and final notional payment from "C": If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that A would face in case "C" defaults.

Fundamental Credit Derivatives: CDS's

"B"	$\rightarrow$	protection $Z =$ "$1 -$ recov" at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$	"A"
"B"	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\leftarrow$	"A"

Usually, at evaluation time (t) the amount $R = R_{a,b}(t)$ is set at a value that makes the contract fair, i.e. such that the present value of the two exchanged flows is zero. This is how the market quotes CDS's: CDS are quoted via their fair R's (Bid and Ask). Formally we may write the (Running) CDS discounted value to "B" at time $t < T_a$ as $\Pi\mathrm{RCDS}_{a,b}(t) :=$

$$D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ Z$$

Accrued rate at default $+$ CDS Rate payments if no default - Protection paym at default

where $u \in [T_{\beta(u)-1}, T_{\beta(u)})$, i.e. $T_{\beta(u)}$ is the first of the T_i's following τ. The stochastic discount factor at time t for maturity T is $D(t,T) = B(t)/B(T)$, where $B(t) = \exp(\int_0^t r_u du)$ is the bank-account numeraire.

Fundamental Credit Derivatives. CDS's: Postponed Payoff

$$\Pi\mathsf{RCDS}_{a,b}(t) = D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ Z$$

Accrued rate at default + CDS Rate payments if no default - Protection paym at default

Consider instead the two POSTPONED PAYMENT alternatives

$$\Pi\mathsf{PRCDS}_{a,b}(t) = \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\geq T_i\}} - \sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{\tau\in(T_{i-1}.T_i]\}}Z$$

$$\Pi\mathsf{PR2CDS}_{a,b}(t) = \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_{i-1}\}} - \sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{\tau\in(T_{i-1}.T_i]\}}Z$$

CDS Rate payments if no default - Protection paym at first T_i after default

The difference between the last two is whether to add a R payment for the date that has been postponed or not. For small ϵ, in paths where $\tau = T_j + \epsilon$ the first postponed payoff is a better approximation, whereas in paths where $\tau = T_j - \epsilon$ the second one is better.

Fundamental Credit Derivatives. CDS's: Risk Neutral Valuation

Denote by CDS$(t, [T_{a+1}, \ldots, T_b], T_a, T_b, R, Z)$, PRCDS$()$, PR2CDS$()$ the time t price of the above Running standard and postponed CDS's. **The pricing formula for this product and its postponed variants depends on the assumptions on interest-rate dynamics and on the default time** τ (reduced form models, structural models...). In general, we can compute the CDS price according to risk-neutral valuation (see e.g. Bielecki and Rutkowski (2002)):

$$\text{CDS}(t, T_a, T_b, R, Z) = \mathbb{E}\{\Pi_{\text{RCDS}_{a,b}} | \mathcal{G}_t\},$$

$$\text{PR(2)CDS}(t, T_a, T_b, R, Z) = \mathbb{E}\{\Pi_{\text{PR(2)CDS}_{a,b}} | \mathcal{G}_t\}$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$, $\mathcal{F}_t$= "info-on-default-free-markets-up-to-t";
$\sigma(\{\tau < u\}, u \leq t) =$ "info if default occurred before t, and, if so, when exactly".
$\mathcal{F}_t$ denotes the basic filtration without default, typically representing the information flow of interest rates, intensities and possibly other default-free market quantities (see Bielecki and Rutkowski (2001)), and $\mathbb{E}$ denotes the risk-neutral expectation in the enlarged probability space supporting τ, the related risk-neutral measure being denoted by $\mathbb{Q}$.
$\mathcal{F}_t = \mathcal{G}_t$ in basic structural models. $\mathcal{F}_t \subset \mathcal{G}_t$ in basic reduced form models.

The Basic Idea of Reduced Form Models

In reduced form or intensity models, the default time τ obeys roughly the following:

Having not defaulted before t, Probability of defaulting in the next dt instants is

$$\text{Prob}(\tau \in [t, t+dt) | \tau > t, \text{market info up to } t) = \lambda(t) dt$$

where the "probability" dt factor λ is called **intensity** or **hazard rate**. It is also an **instantaneous credit spread** (more on this later). Intensity can be

- Constant (τ is first jump of time homogeneous Poisson process);
- Time varying (τ is first jump of time inhomogeneous Poisson process); Can model the term structure of credit spreads; Does not model credit spread volatility; Implied hazard functions;
- Stochastic (τ is first jump of Cox Process); Can model term structure of credit spreads; Can model credit spread volatility;

Modeling Tools: Poisson processes

Time homogeneous Poisson process M_t with (constant) intensity $\bar{\gamma}$: It is a unit-jump increasing, right continuous process with stationary independent increments and $M_0 = 0$. If further $\bar{\gamma} = 1$ the process is called Standard Poisson Process (SPP). We know that

$$M_t - M_s \sim \mathcal{P}((t-s)\bar{\gamma}), \quad \mathbb{Q}\{M_t - M_s = k\} = e^{-\bar{\gamma}(t-s)}(\bar{\gamma}(t-s))^k/k!$$

where $\mathcal{P}$ is the Poisson law, and $M_t - M_s$ is independent of $\sigma(\{M_u, u \leq s\})$.
Assume for a moment that the first jump time $\bar{\tau}$ of M_t is the default time. It has the following properties: $\bar{\gamma} \cdot \bar{\tau} \sim$ exponential(1), independent of $\mathcal{F}$, and we have

$$\mathbb{Q}\{\bar{\tau} \in [t, t+dt) | \bar{\tau} \geq t\} = \bar{\gamma}\ dt :$$

"probability that company defaults in (arbitrarily small) "dt" years given that it has not defaulted so far is $\bar{\gamma}\ dt$." Also, prob of defaulting between s and t is

$$\mathbb{Q}\{s < \bar{\tau} \leq t\} = \exp(-\bar{\gamma}s) - \exp(-\bar{\gamma}t) \approx \bar{\gamma}(t-s)$$

Modeling Tools: Cox processes

Models with **time-varying deterministic intensity**: In these models the default time τ is the 1st jump-time of a **time-inhomogeneous Poisson Process** (PP) N_t with increasing, continuous (to simplify life, so we have invertibility, but this would not be necessary) hazard function Γ and hazard rate (deterministic intensity) $\gamma(t)$, with $\Gamma(T) = \int_0^T \gamma(t)dt$.

Intensity can be also time-varying and **stochastic**: in that case it is assumed to be at least a $\mathcal{F}_t$-adapted and right continuous (and thus progressive) process and is denoted by λ_t and $\Lambda(T) = \int_0^T \lambda_t dt$. In this case, conditional on $\mathcal{F}^\lambda$, we still have a PP structure. Under stochastic intensity, the final process jumping first at τ is called a **Cox process.**

We have $\mathbb{Q}\{\tau \in [t, t+dt) | \tau \geq t, \mathcal{F}_t\} = \lambda_t \, dt$. This reads, if "$t$=now":
"probability that company defaults in (arbitrarily small) "dt" years given that it has not defaulted so far and given the market information so far is $\lambda_t \, dt$."

Modeling Tools: time inhomogeneous Poisson Processes

Back to **deterministic time-varying** intensity $\gamma(t)$. PP theory tells us $\mathbf{N_t} = \mathbf{M_{\Gamma(t)}}$ (or $M_t = N_{\Gamma^{-1}(t)}$) where M_t is SPP. **Thus a PP is just a time-changed SPP** M.

Since $N_t = M_{\Gamma(t)}$, if N jumps first at τ, then M jumps first at $\Gamma(\tau) = \int_0^\tau \gamma(u)du$.

But since M is Poisson with intensity one, its first jump time $\Gamma(\tau)$ is known to be an exponential random variable with parameter 1, so that ($\Gamma(\tau) = \xi \sim$ exponential(1)) $\mathbb{Q}\{\Gamma(\tau) < s\} = 1 - \exp(-s)$. In particular, since Γ is strictly increasing,

$$\mathbb{Q}\{s < \tau \leq t\} = \mathbb{Q}\{\Gamma(s) < \Gamma(\tau) \leq \Gamma(t)\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)) \text{ i.e.}$$

"prob of default between s and t is "$\mathbf{e^{-\int_0^s \gamma(u)du} - e^{-\int_0^t \gamma(u)du} \approx \int_s^t \gamma(u)du}$" (where the final approximation is good for small exponents).

Modeling Tools: time inhomogeneous Poisson Processes

Recall that from Poisson Process Theory $\Gamma(\tau) = \xi \sim$ exponential(1). Then

$$\mathbb{Q}\{\tau \geq s\} = \mathbb{Q}\{\Gamma(\tau) \geq \Gamma(s)\} = \mathbb{Q}\left\{\xi \geq \int_0^s \gamma(u)du\right\} = \exp\left(-\int_0^s \gamma(u)du\right)$$

which is the complete analogous of a discount factor where γ is the instantaneous interest rate. This is why γ can be interpreted as a **instantaneous credit spread**.

With stochastic intensity, under standard assumptions one can show $\mathbb{Q}\{\tau \geq s\} =$

$$= \mathbb{Q}\{\Lambda(\tau) \geq \Lambda(s)\} = \mathbb{Q}\left\{\xi \geq \int_0^s \lambda(u)du\right\} = \mathbb{E}\left[\mathbb{Q}\left\{\xi \geq \int_0^s \lambda(u)du \middle| \mathcal{F}^\lambda\right\}\right] = \mathbb{E}\left[e^{-\int_0^s \lambda(u)du}\right]$$

which is completely analogous to the bond price formula in a short rate model with interest rate λ. **Cox processes allow to drag the interest-rate technology and paradigms into default modeling**. But...

ξ is independent of all default free market quantities and represents an external source of randomness that makes reduced form model incomplete.

Modeling Tools: Cox processes

Example of $T \mapsto \mathbb{Q}\{\tau \leq T\} = 1 - \exp(-\int_0^T \gamma(u)du) = 1 - e^{-\Gamma(T)} \approx \Gamma(T)$

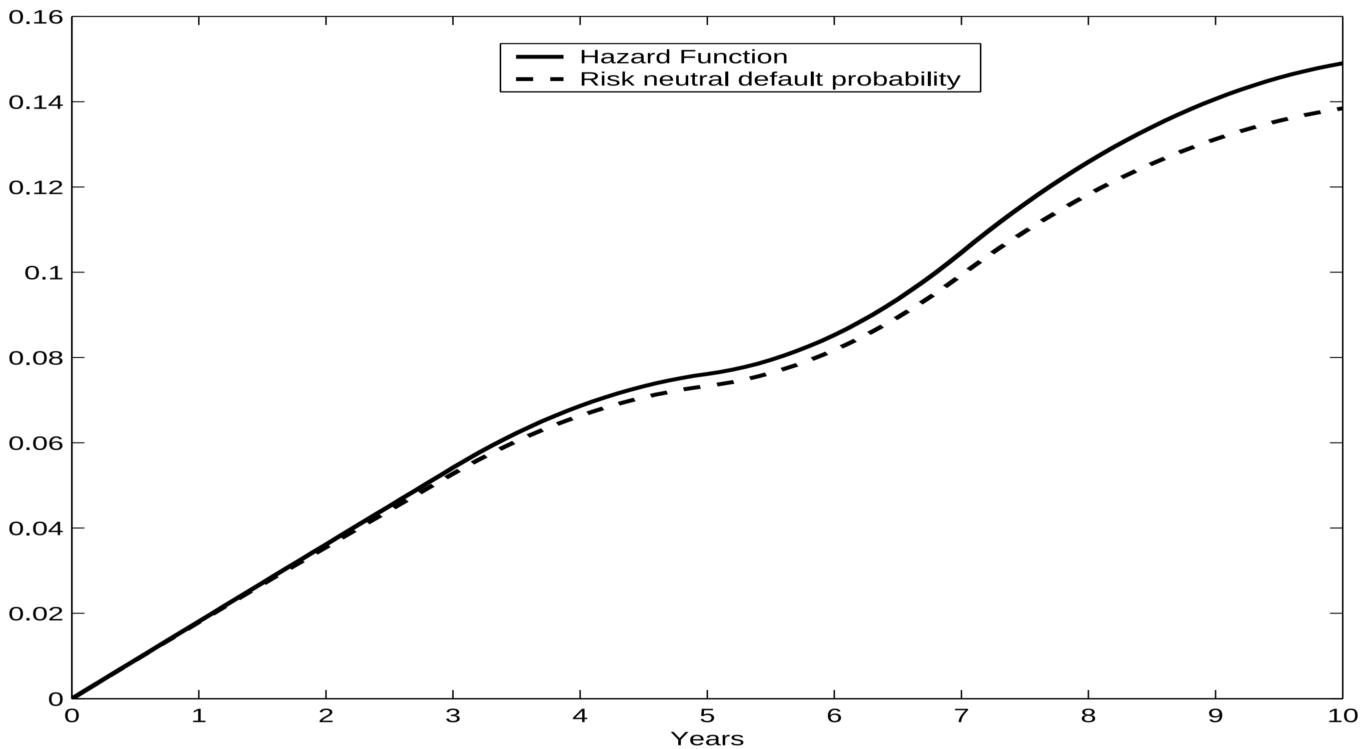

Figure 1: Hazard function and risk-neutral default probability for Merrill-Lynch CDS's, October 25, 2002

Modeling Tools: Cox processes

Summing up:

SPP : Standard Poisson Processes (with unit constant intensity, i.e. instantaneous jump probability) are the probabilistic basis; **Intensity** (or instantaneous credit spread) **is constant** and set to one.

PP : Time inhomogeneous Poisson processes can be built based on SPP and on a given **deterministic time-varying intensity**; these are often used as a quoting mechanism for credit spreads in **CDS and Corporate Bond** contracts, the intensity being also interpreted as an **instantaneous credit spread**;

COX : If the intensity is Stochastic, conditional on the intensity filtration we have a PP and this is a Cox process. These models can be used for more sophisticated credit derivatives and take into account also **credit spread volatility.**

Two Important Technicalities

First: If we assume interest rates to be stochastic and to be driven by Brownian Motions as sources of randomness, since a Poisson process and a Brownian motion defined on the same probability space are independent (see for example Bielecki and Rutkowski (2001), p. 188), the processes N and r are independent.
We can thus assume the stochastic discount factor $D(s,t) = \exp(-\int_s^t r_u du)$*, and the default time* τ *to be* **independent** *under deterministic intensities* for τ

Second: A Fundamental Technical Result by Jeanblanc and Rutkowski (JR).
In a Cox process setting, where we assume an $\mathcal{F}_t$ progressively measurable and positive stochastic intensity λ with integrable paths, under very general measurability conditions for the payoff (typically the payoff is assumed to be $\mathcal{G}_\infty$-measurable) and for $t < T$ we have

$$\mathbb{E}(1_{\{\tau>T\}}\text{Payoff}|\mathcal{G}_t) = \frac{1_{\{\tau>t\}}}{\mathbb{Q}\{\tau > t|\mathcal{F}_t\}}\mathbb{E}(1_{\{\tau>T\}}\text{Payoff}|\mathcal{F}_t)$$

Recall: $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$, $\mathcal{F}_t$="info-on-default-free-markets-up-to-t";
$\sigma(\{\tau < u\}, u \leq t) =$"info if default occurred before t, and, if so, when exactly".
Switching from $\mathcal{G}$ expectations to $\mathcal{F}$ expectations is important because for some variables the $\mathcal{F}$ conditional expectations are easier to compute.

CDS rates and CDS forward rates

$$\text{(PR(2))CDS}(t, T_a, T_b, R, Z) = \boxed{\mathbb{E}\{\Pi_{a,b}|\mathcal{G}_t\} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\{\Pi_{a,b}|\mathcal{G}_t\}} = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{F}_t)}\boxed{\mathbb{E}\{\Pi_{a,b}|\mathcal{F}_t\}}$$

with $\Pi_{a,b} = \Pi\text{RCDS}_{a,b}(t)$, $\Pi_{a,b} = \Pi\text{PRCDS}_{a,b}(t)$ and $\Pi_{a,b} = \Pi\text{PR2CDS}_{a,b}(t)$, i.e. the $\text{CDS}_{a,b}$ discounted payoffs and its postponed variants respectively. The CDS forward rate $R_{a,b}(t)$ is defined as R satisfying $\text{CDS}(t, T_a, T_b, R, Z) = 0.$ "Second box=0" gives:

$$R_{a,b}(t) = \frac{Z\ \mathbb{E}[D(t,\tau)\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i) + \mathbb{E}\left\{D(t,\tau)(\tau - T_{\beta(\tau)-1})\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t\right\}},$$

$$R^P_{a,b}(t) = \frac{Z\ \sum_{i=a+1}^{b}\mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)},\quad R^{P2}_{a,b}(t) = \frac{Z\ \sum_{i=a+1}^{b}\mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}|\mathcal{F}_t]}$$

where

$$\mathbf{1}_{\{\tau>t\}}\bar{\mathbf{P}}(\mathbf{t},\mathbf{T}) := \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t)$$

CDS rates and CDS forward rates

$$\text{(PR(2))CDS}(t, T_a, T_b, R, Z) = \boxed{\mathbb{E}\{\Pi_{a,b}|\mathcal{G}_t\} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\{\Pi_{a,b}|\mathcal{G}_t\}} = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{F}_t)}\boxed{\mathbb{E}\{\Pi_{a,b}|\mathcal{F}_t\}}$$

with $\Pi_{a,b} = \Pi\text{RCDS}_{a,b}(t)$, $\Pi_{a,b} = \Pi\text{PRCDS}_{a,b}(t)$ and $\Pi_{a,b} = \Pi\text{PR2CDS}_{a,b}(t)$.
The CDS forward rate $R_{a,b}(t)$ is defined as R satisfying $\text{CDS}(t, T_a, T_b, R, Z) = 0$. "Second box=0" gives the above definitions of $R_{a,b}$, $R^P_{a,b}$, $R^{P2}_{a,b}$. In these definitions

$$\mathbf{1}_{\{\tau>t\}}\bar{\mathbf{P}}(\mathbf{t}, \mathbf{T}) := \mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}[D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]/\mathbb{Q}(\tau > t|\mathcal{F}_t)$$

is the price at time t of a **defaultable zero coupon bond** maturing at time T, while we will denote by $\mathbf{P}(\mathbf{t}, \mathbf{T})$ the corresponding default free bond.
These definitions of R are possible on all trajectories and the denominator (candidate numeraire) of $R_{a,b}$ does not vanish in any trajectory. This is not the case if we derive R by imposing "first box=0": in that case, R is defined only on the paths $\{\tau > t\}$.

CDS rates and CDS forward rates

Take for example $R^P_{a,b}$. We have

$$R^P_{a,b}(t) = \frac{Z\ \sum_{i=a+1}^b \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^b \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)}.$$ Define $$R^P_i(t) = \frac{Z\ \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)}$$

Actually R^P_i is the rate $R^P_{i-1,i}(t)$ of a particular postponed CDS with only one payment date (similar to the forward LIBOR rate making FRA contracts fair). Notice that we can write

$$R_{a,b}(t) = \sum_{j=a+1}^b \bar{w}_j(t) R_j(t) \approx \sum_{j=a+1}^b \bar{w}_j(0) R_j(t), \quad \bar{w}_j(t) = \frac{\bar{P}(t,T_j)}{\sum_{i=a+1}^b \alpha_i \bar{P}(t,T_i)},$$

a weighted average, analogous to expressing the swap rate as an average of forward LIBOR rates in the default free market.

CDS forward rates and defaultable LIBOR rates

Consider fwd defaultable rates "$\bar{F}_j(t) := (1/\alpha_j)(\bar{P}(t,T_{j-1})/\bar{P}(t,T_j) - 1)$ on $\tau > t$" (Schönbucher, mimics the default free case).
Consider a one-period CDS with $T_a = T_{j-1}$ and $T_b = T_j$. We obtain (take $Z = 1$)

$$R_j^P(t) := \frac{\mathbb{E}[D(t,T_j)\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)} = \frac{\mathbb{E}[D(t,\boxed{T_j})\mathbf{1}_{\{\tau>T_{j-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_j)\mathbf{1}_{\{\tau>T_j\}}|\mathcal{F}_t]}{\alpha_j \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_j)}$$

The analogous part of $\bar{F}_j$ would be the same but with T_{j-1} replacing the boxed T_j. The difference is that in R_j^P we take expectation of a quantity that vanishes for all paths ω where $\tau > T_j$, whereas in $\bar{F}$ the quantity inside the $\mathbb{E}$ does not vanish for $\tau > T_j$.
Schönbucher (2000) defines the discrete tenor credit spread, in general, to be

$$H_j(t) := \frac{1}{\alpha_j}\left(\frac{\bar{P}(t,T_{j-1})/P(t,T_{j-1})}{\bar{P}(t,T_j)/P(t,T_j)} - 1\right)$$

(this definition is valid only in $\tau > t$), and if intensities and interest rates are independent we get $H_i(t) = R_i^P(t)$ and $R_i^P(t) \approx \frac{1}{T_j - T_{j-1}}\int_{T_{j-1}}^{T_j} \gamma(s)ds$ (average credit spread).

CDS valuation with determ intensities: Implied intensity γ and Γ

$$\text{CDS}(t, T_a, T_b, R, Z; \Gamma(\cdot)) = \mathbf{1}_{\{\tau > t\}} \Bigg[R \int_{T_a}^{T_b} P(t,u)(T_{\beta(u)-1} - u) d(e^{-(\Gamma(u)-\Gamma(t))})$$

$$+ \sum_{i=a+1}^{b} P(t,T_i) R \alpha_i e^{-(\Gamma(T_i)-\Gamma(t))} + Z \int_{T_a}^{T_b} P(t,u) d(e^{-(\Gamma(u)-\Gamma(t))}) \Bigg]$$

This holds when τ is first jump time of PP with intensity γ and hazard function $\Gamma(t) = \int_0^t \gamma(u)du$. Market Quoting Mechanism: **The market quotes fair R^{mkt}'s that make the CDS value equal to zero**. One may wish to **calibrate** the model's Γ to such prices to value different payoffs. Find the Γ^{mkt}'s solving (for several T_b's)

$$\text{CDS}(0, 0, T_b, R_{0,b}^{\text{mkt MID}}(0), Z; \Gamma^{\text{mkt}}(\cdot)) = 0$$

If we are given $R_{0,b}^{\text{mkt MID}}(0)$ for different maturities T_b, we can assume a *piecewise linear* (or at times constant) γ, and invert prices in an iterative way as T_b increases, deriving each time the new part of γ that is consistent with the R for the new increased maturity.

CDS's with deterministic intensities: Example

Piecewise quadr $\Gamma(t)$ and related prob ($\mathbb{Q}\{\tau < t\} = 1 - \exp(-\Gamma(t)) \approx \Gamma(t)$ for small Γ) obtained by calibrating the 1y, 3y, 5y, 7y and 10y CDS's on Merrill-Lynch

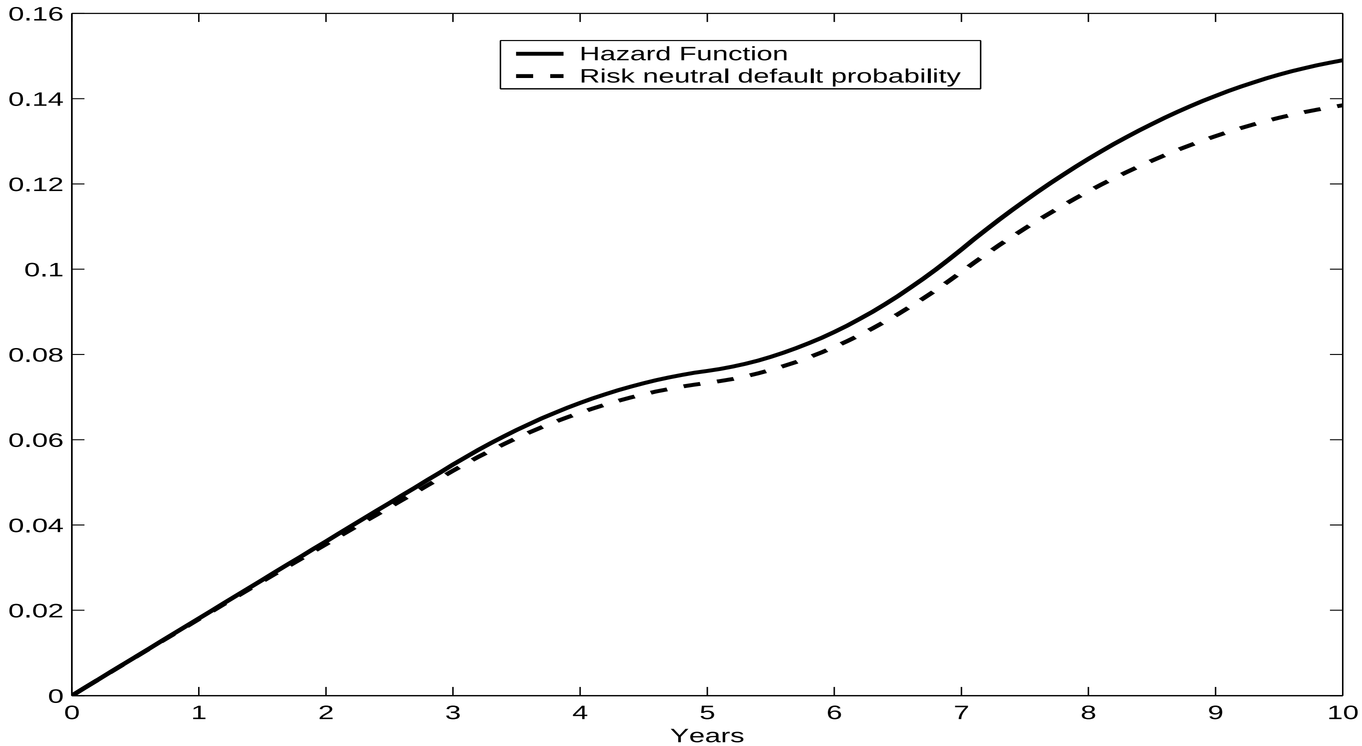

Figure 2: Hazard function and risk-neutral default probability for Merrill-Lynch CDS's, October 25, 2002

Fundamental Credit Derivatives: Defaultable Floaters

Prototypical defaultable floating-rate note ensures the payment at $T_{a+1}, \ldots, T_b$ of the LIBOR rates that reset at the previous instants $T_a, \ldots, T_{b-1}$ plus a spread X, plus the notional at final T_b, each payment conditional on the issuer having not defaulted before the relevant previous instant. We assume a deterministic recovery value S to be paid at the first T_i following default if default occurs before T_b.
Recall that without default the fair spread making the FRN quote at par is 0.
With Default, the note discounted payoff, including the notional invested in T_a, is

$$\Pi\text{DFRN}_{a,b} = -D(t, T_a)1_{\{\tau > T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t, T_i)(L(T_{i-1}, T_i) + X)\mathbf{1}_{\{\tau > T_i\}}$$

$$+D(t, T_b)1_{\{\tau > T_b\}} + S \sum_{i=a+1}^{b} D(t, T_i)\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}},$$

where S is the recovery rate, i.e. the percentage of the notional that is paid in replacement of the notional in case of default, and it is paid at the first instant among $T_{a+1}, \ldots, T_b$ following default. This is the correct definition of DFRN, consistent with market practice.

Fundamental Credit Derivatives: Defaultable Floaters

$$\Pi\mathsf{DFRN}_{a,b} = -D(t,T_a)1_{\{\tau > T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)(L(T_{i-1},T_i) + X) \boxed{\mathbf{1}_{\{\tau > T_i\}}}$$

$$+D(t,T_b)1_{\{\tau > T_b\}} + S \sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{T_{i-1} < \tau \le T_i\}},$$

The problem with such definition is that it has no good equivalent in terms of CDS payoff. This is due to the fact that, in a Cox process setting, it is difficult to disentangle the LIBOR rate L from the indicator and stochastic discount factor in such a way to obtain expectations of pure stochastic discount factors times default indicators. This becomes possible if we replace $1_{\{\tau > T_i\}}$ in the first summation with $1_{\{\tau > T_{i-1}\}}$. We thus consider

$$\Pi\mathsf{DFRN2}_{a,b} = -D(t,T_a)1_{\{\tau > T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)(L(T_{i-1},T_i) + X) \boxed{\mathbf{1}_{\{\tau > T_{i-1}\}}} + \ldots$$

Fundamental Credit Derivatives: Defaultable Floaters

$$\Pi\text{DFRN2}_{a,b} = -D(t,T_a)1_{\{\tau > T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t,T_i)(L(T_{i-1},T_i) + X) \boxed{\mathbf{1}_{\{\tau > T_{i-1}\}}}$$

$$+D(t,T_b)1_{\{\tau > T_b\}} + S \sum_{i=a+1}^{b} D(t,T_i)\mathbf{1}_{\{T_{i-1} < \tau \le T_i\}},$$

Recall that, in the CDS payoff, protetion= $Z = 1 - S$. We may now value the above discounted payoff at time t and derive the value of X that makes it 0. Define

$$\text{DFRN2}_{a,b}(t,X,S) = \mathbb{E}\{\Pi\text{DFRN2}_{a,b}|\mathcal{G}_t\} = \mathbf{1}_{\{\tau > t\}}\mathbb{E}\{\Pi\text{DFRN2}_{a,b}|\mathcal{F}_t\}/\mathbb{Q}(\tau > t|\mathcal{F}_t)$$

and solve $\mathbb{E}\{\Pi\text{DFRN2}_{a,b}|\mathcal{F}_t\} = 0$ in X. The only nontrivial part is computing

$$\alpha_i\mathbb{E}[D(t,T_i)L(T_{i-1},T_i)\mathbf{1}_{\{\tau > T_{i-1}\}}|\mathcal{F}_t] = \mathbb{E}[D(t,T_{i-1})\mathbf{1}_{\{\tau > T_{i-1}\}}|\mathcal{F}_t] - \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau > T_{i-1}\}}|\mathcal{F}_t]$$

Now the LIBOR flow has vanished from the above payoff and we have expressed everything in terms of pure discount factor and default indicators. This is not possible with the original $\Pi\text{DFRN}_{a,b}$

Fundamental Credit Derivatives: Defaultable Floaters

Through explicit computations, we find that

$$\mathsf{DFRN2}_{a,b}(t,X,S) = \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\Big[-Z\sum_{i=a+1}^{b}\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}]$$

$$+X\sum_{i=a+1}^{b}\alpha_i\mathbb{E}_t[D(t,T_i)\mathbf{1}_{\{\tau>T_{i-1}\}}]\Big],$$

from which we notice that

$$\mathsf{DFRN2}_{a,b}(t,X,S) = \mathsf{PR2CDS}(t,T_a,T_b,X,1-S). \tag{1}$$

By taking into account this result, the expression for X that makes the DFRN quote at par is clearly the running "postponed of the second kind" CDS forward rate

$$X^{(2)}_{a,b}(t) = R^{P2}_{a,b}(t),$$

i.e. *the fair spread in a defualtable floating rate note is equal to the running postponed CDS forward rate*.

Fundamental Credit Derivatives: Defaultable Floaters

The second alternative definition of DFRN, leading to a useful relationship with approximated CDS payoffs, is obtained by moving the default indicator of $L(T_{i-1}, T_i) + X$ from T_i to T_{i-1} but only for the LIBOR flow, not for the spread X. This payoff is closer to the original ΠDFRN payoff than the approximated ΠDFRN2 payoff considered above. Set

$$\Pi\text{DFRN1}_{a,b} = -D(t, T_a)1_{\{\tau > T_a\}} + \sum_{i=a+1}^{b} \alpha_i D(t, T_i)(L(T_{i-1}, T_i) \boxed{\mathbf{1}_{\{\tau > T_{i-1}\}}} + X\mathbf{1}_{\{\tau > T_i\}})$$

$$+D(t, T_b)1_{\{\tau > T_b\}} + S \sum_{i=a+1}^{b} D(t, T_i)\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}}.$$

By calling $\text{DFRN1}_{a,b}(t, X, S)$ the t-value of the above payoff and by going through the computations we can see easily that this time

$$\text{DFRN1}_{a,b}(t, X, S) = \text{PRCDS}(t, T_a, T_b, X, 1 - S). \tag{2}$$

and that, as far as fair spreads are concerned, $X^{(1)}_{a,b}(t) = R^{PR}_{a,b}(t)$.

CDS Options and Callable Defaultable Floaters

Consider the option to enter a CDS at a future time $T_a > 0$, $T_a < T_b$, paying a fixed rate K at times $T_{a+1}, \ldots, T_b$ or until default, in exchange for a protection payment Z against possible default in $[T_a, T_b]$ (payer CDS option). If default occurs Z is received. By noticing that the market CDS rate $R_{a,b}(T_a)$ will set the CDS value in T_a to 0, the payoff can be written as the discounted difference between said CDS and the corresponding CDS with rate K. We will see below that this is equivalent to a call option on the future CDS fair rate $R_{a,b}(T_a)$. The discounted CDS option payoff reads, at time t,

$$\Pi_{\text{CallCDS}_{a,b}}(t;K) = D(t,T_a)[\text{CDS}(T_a,T_a,T_b,R_{a,b}(T_a),Z) - \text{CDS}(T_a,T_a,T_b,K,Z)]^+,$$

so that, by writing the CDS expressions explicitly

$$\begin{aligned}\Pi_{\text{CallCDS}_{a,b}}(t;K) &= \frac{\mathbf{1}_{\{\tau > T_a\}}}{\mathbb{Q}(\tau > T_a|\mathcal{F}_{T_a})} D(t,T_a)\left[\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}(\tau > T_a|\mathcal{F}_{T_a})\bar{P}(T_a,T_i) + \right. \\ &+ \left. \mathbb{E}\left\{D(T_a,\tau)(\tau - T_{\beta(\tau)-1})\mathbf{1}_{\{\tau < T_b\}}|\mathcal{F}_{T_a}\right\}\right](R_{a,b}(T_a) - K)^+\end{aligned}$$

CDS Options and Callable Defaultable Floaters

$$\Pi\text{CallCDS}_{a,b}(t;K) = \frac{1_{\{\tau>T_a\}}}{\mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})} D(t,T_a)\left[\sum_{i=a+1}^{b}\alpha_i \mathbb{Q}(\tau>T_a|\mathcal{F}_{T_a})\bar{P}(T_a,T_i)+\right.$$

$$\left. + \quad \mathbb{E}\left\{D(T_a,\tau)(\tau-T_{\beta(\tau)-1})\mathbf{1}_{\{\tau<T_b\}}|\mathcal{F}_{T_a}\right\}\right](R_{a,b}(T_a)-K)^+$$

These options can be introduced also for postponed CDS. We will often neglect the $(\tau - T_{\beta(\tau)-1})$ term. In such a case the quantity [.] is called ("no survival-indicator"-) "defaultable present value per basis point (DPVBP) numeraire". Actually the real DPVBP would have a $1_{\{\tau> \cdot\}}$ term in front of the summation. More generally, at time t, we set

$$\widehat{C}_{a,b}(t) := \mathbb{Q}(\tau>t|\mathcal{F}_t)\bar{C}_{a,b}(t), \quad \bar{C}_{a,b}(t) := \sum_{i=a+1}^{b}\alpha_i\bar{P}(t,T_i).$$

When including as a factor the indicator $1_{\{\tau>t\}}$, this quantity is the price, at time t, of a portfolio of defaultable zero-coupon bonds with zero recovery and with different maturities, and as such it is the price of a tradable asset, hence a possible numeraire.

CDS Options and Callable Defaultable Floaters

Neglecting the accrued interest term, the payoff simplifies to the **approximated one**

$$1_{\{\tau>T_a\}} D(t,T_a) \left[\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a,T_i)\right] (R_{a,b}(T_a) - K)^+$$

First equivalence CDS/FLOATERS OPTIONS: PRCDS and DFRN1. Let us follow the same derivation under the postponed CDS payoff of the first kind. Consider thus

$$\Pi_{\text{CallPRCDS}_{a,b}}(t;K) = D(t,T_a)[\text{PRCDS}(T_a,T_a,T_b,R^{PR}_{a,b}(T_a),Z) - \text{PRCDS}(T_a,T_a,T_b,K,Z)]^+$$

or, given our earlier equivalence result,with $Z = 1 - S$,

$$D(t,T_a)[\text{DFRN1}_{a,b}(T_a, X_{a,b}(T_a), S) - \text{DFRN1}_{a,b}(T_a,K,S)]^+.$$

By expanding the expression of PRCDS we obtain as *exact* discounted payoff the quantity

$$\Pi_{\text{CallPRCDS}_{a,b}}(t,K) = 1_{\{\tau>T_a\}} D(t,T_a) \left[\sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a,T_i)\right] (R^{P}_{a,b}(T_a) - K)^+.$$

CDS Options and Callable Defaultable Floaters

SECOND equivalence CDS/FLOATERS OPTIONS: PR2CDS and DFRN2. We may also consider the postponed running CDS of the second kind. The related discounted CDS option payoff reads, at time t,

$$\Pi_{\text{CallPR2CDS}_{a,b}}(t, K) = D(t, T_a)[\text{PR2CDS}(T_a, T_b, R^{P2}_{a,b}(T_a), Z) - \text{PR2CDS}(T_a, T_b, K, Z)]^+,$$

and given (1), this is equivalent to

$$D(t, T_a)[\text{DFRN2}_{a,b}(T_a, X_{a,b}(T_a), S) - \text{DFRN2}_{a,b}(T_a, K, S)]^+,$$

with $S = 1 - Z$, or, by expanding the expression for PR2CDS, as

$$(1_{\{\tau > T_a\}}/\mathbb{Q}(\tau > T_a|\mathcal{F}_{T_a}))D(t, T_a) \sum_{i=a+1}^{b} \alpha_i \mathbb{E}_{T_a}[D(T_a, T_i)\mathbf{1}_{\{\tau > T_{i-1}\}}](R^{PR2}_{a,b}(T_a) - K)^+.$$

Again we have equivalence between CDS options and options on the defaultable floater, but the numeraire here is more delicate

CDS Options: Market Models (embedded Stochastic Intensity)

As usual, one may wish to introduce implied volatility for CDS options. This would be a volatility associated to the relevant underlying CDS rate R.
In order to do so rigorously, one has to come up with an appropriate dynamics for $R_{a,b}$ directly, rather than modeling instantaneous default intensities explicitly. This is done in a Cox process setting where we do not model directly stochastic intensity but rather some secondary market quantities that depend on it.
This parallels the default-free interest rate market when we resort to the swap market model as opposed for example to a one-factor short-rate model for pricing swaptions.
In the case of CDS options the market model is derived as follows. Consider as example the PR1CDS formulation. Take as numeraire the DPVBP $\widehat{C}_{a,b}$, so that

$$R^P_{a,b}(t) = \frac{Z\ \sum_{i=a+1}^b \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^b \alpha_i \mathbb{Q}(\tau > t|\mathcal{F}_t)\bar{P}(t,T_i)} = \frac{Z\ \sum_{i=a+1}^b \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}|\mathcal{F}_t]}{\widehat{C}_{a,b}(t)},$$

$t \le T_a$ is a martingale under this measure and can be modeled as a Black-Scholes (BS) driftless geometric Brownian motion, leading to a BS formula for CDS options.

CDS Options: Market Models (embedded Stochastic Intensity)

We denote here R^P by R. Compute, by resorting to the change of numeraire, iterated conditioning and switching from $\mathcal{G}$ to $\mathcal{F}$ (Payer CDS option):

$$\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)(R_{a,b}(T_a)-K)^+|\mathcal{G}_t\} = \ldots$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\mathbb{E}\big[D(t,T_a)\widehat{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+|\mathcal{F}_t\big]$$

$$= \frac{1_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{F}_t)}\widehat{C}_{a,b}(t)\widehat{\mathbb{E}}^{a,b}[(R_{a,b}(T_a)-K)^+|\mathcal{F}_t] = 1_{\{\tau>t\}}\bar{C}_{a,b}(t)\widehat{\mathbb{E}}^{a,b}[(R_{a,b}(T_a)-K)^+|\mathcal{F}_t]$$

and we may take (Jamshidian (2002), Brigo (2003))

$$dR_{a,b}(t) = \sigma_{a,b}R_{a,b}(t)dW^{a,b}(t),$$

where $W^{a,b}$ is a Brownian motion under $\widehat{\mathbb{Q}}^{a,b}$, leading to a market formula

$$\mathbb{E}\{1_{\{\tau>T_a\}}D(t,T_a)\bar{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+|\mathcal{G}_t\} = 1_{\{\tau>t\}}\bar{C}_{a,b}(t)[R_{a,b}(t)N(d_1(t))-KN(d_2(t))]$$

$$d_{1,2} = \left(\ln(R_{a,b}(t)/K) \pm (T_a-t)\sigma_{a,b}^2/2\right)/(\sigma_{a,b}\sqrt{T_a-t}).$$

CDS Options: Market Models (embedded Stochastic Intensity)

As happens in most markets, this Black-like formula could be used as a implied volatility quoting mechanism rather than as a real model formula.

Furthermore, the numeraire martingale framework is general and can include CDS Options smiles that may arise in the market in the coming years. Indeed, we are not forced to take

$$dR_{a,b}(t) = \sigma_{a,b} R_{a,b}(t) dW^{a,b}(t),$$

but can also assume

$$dR^{PR}_{a,b}(t) = \nu_{a,b}(t, R^{PR}_{a,b}(t)) R^{PR}_{a,b}(t) dW^{a,b}(t)$$

with ν a suitable deterministic function of time and state. We might choose the CEV dynamics, a displaced diffusion dynamics, a hyperbolic sine densities mixture dynamics or a lognormal mixture dynamics. Several tractable choices are possible already in the local volatility diffusion setup, and one may select a smile dynamics for the LIBOR or swap model and use it to model R. There are several possible choices. For example, one may select $\nu_{a,b}$ from Brigo and Mercurio (2003) or Brigo Mercurio and Sartorelli (2003).

CDS Options and Callable Defaultable Floaters

Examples of CDS implied volatilities (Model implementation by Marco Tarenghi).

3 European Telephone companies C1 (Baa2/BBB+), C2 (A2/A), C3 (Baa2/BBB+).

Data as of march 10, 2004; Recovery $= 0.5$; $Z = 1 - 0.5 = g0.5$;

$T_0 = 0 =$ march 10 2004; $T_a =$June 20 04 (3m10d); $T_b =$ march 20 09 (5y10d);

Receiver option quotes (puts on R) in basis points (i.e. 1E-4 units on a notional of 1)

	bid	mid	ask (bps)	$R_{0,b}(0)$	$R_{a,b}(0)$	K (bps)	mid $\sigma_{a,b}$
C1	25	32.5	40	60	61	60	62.16%
C2	17	24.5	32	43	43.4	43	63.71%
C3	25	32.5	40	61	62	61	61.46%

Implied volatilities are rather high when compared with interest-rate default free swap volatilities (typically below 20%).

Explicit Stochastic Intensity : The SSRD model

Consider the following model for stochastic interest rates r and intensities λ, "shifted square root diffusion model" (SSRD) (introduced in Brigo - Alfonsi (2002)):

$$\begin{aligned} dx_t &= k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k, \theta, \sigma, x_0), \quad 2k\theta > \sigma^2, \\ dy_t &= \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t, \ \beta = (\kappa, \mu, \nu, y_0), \ 2\kappa\mu > \nu^2, \ \ dZ\, dW = \rho\, dt \end{aligned}$$

$$\boxed{r_t = x_t^\alpha + \phi(t;\alpha), \quad \lambda_t = y_t^\beta + \psi(t;\beta)}.$$

The quantities ϕ and ψ are **deterministic shifts** that can be set so as to calibrate market interest rates and implied intensities curves: if P^{CIR} is the bond formula in the CIR model,

$$\int_0^t \phi(u,\beta)du = \ln P^{\text{CIR}}(0,t;x_0,\alpha) - \ln P^{\text{Mkt}}(0,t), \int_0^t \psi(u,\beta)du = \ln P^{\text{CIR}}(0,t;y_0,\beta) + \Gamma^{\text{Mkt}}(t)$$

The r model has analytical formulas for zero curve and caps; The λ model has formulas for $\mathbb{Q}$ default probabilities; The models are correlated (ρ). **This is the only known stochastic intensity model with positive intensities and analytical CDS calibration**.

SSRD model: Separable Calibration Interest Rates / Credit

Calibration : r **calibrates** the market zero-coupon curve $T \mapsto P^{\text{MKT}}(0,T)$ analytically by inputing a transformation of said curve in part of ϕ, and cap quotes through calibration of the cap market based on the analytical caps formula in α (Brigo-Mercurio (2001,2001b));

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t, \quad \lambda_t = y_t^\beta + \psi(t;\beta), \quad \beta = (\kappa,\mu,\nu,y_0), \quad \text{corr}(dr,d\lambda) = \rho$$

λ **calibrates** the CDS's implied hazard function $T \mapsto \Gamma^{\text{mkt}}(T)$ by inputing a transformation of this curve in part of ψ analytically (Brigo Alfonsi (2002)), while β is found by minimizing $\int_0^{T_b} \psi^2(u;\beta)du$ (i.e. by keeping the calibrating shift to a minimum, but in coming years β's can be used to calibrate to liquid CDS options quotes); This **separate calibration** implicitly assumes $\rho = 0$ in the SSRD CDS discounted payoff expectation. Check **impact of** ρ on CDS's: we calibrate the model with $\rho = 0$, then set a $\rho \neq 0$ and reprice the same CDS, checking the corresponding price change.

SSRD model: Separable Calibration Interest Rates / Credit

Calibration on a concrete case (details can be skipped at first reading).

Market quotes $R_{0,5}^{\mathsf{BID}} = .009$, $R_{0,5}^{\mathsf{ASK}} = .0098$, $R_{0,5}^{\mathsf{MID}} = .0094$, $Z = .593$ ($R_{0,5}^{\mathsf{MID}}$ rendering the initial 5y CDS fair, $\mathrm{CDS}(0, 0y, 5y, R_{0,5}^{\mathsf{MID}}, Z) = 0$);

Extract implied γ's from R's as explained above. We obtain γ^{mkt}'s of earlier plot.

r's (α, ϕ) come from a typical calibration to interest rates,

(β, ψ) are calibrated to the extracted Γ^{mkt} (earlier plot) and minimizing $\int_0^5 \psi^2(u; \beta)du$.

Recall that this procedure is correct only for $\rho = 0$.

Results: $\kappa = 0.354201,\ \mu = 0.00121853,\ \nu = 0.0238186;\ y_0 = 0.0181$ $(\alpha :\ \ k = 0.528905,\ \theta = 0.0319904,\ \ \sigma = 0.130035,\ x_0 = 8.32349 \times 10^{-5})$.

SSRD model: the impact of correlation

By Monte Carlo simulation of the joint CIR processes with the found α and β and with 140000 paths (and control variate techniques) we obtain

CDS prices	Gaussian Mapping	Monte Carlo value and 95% window
$\rho = -1$	-1.12E-4	-1.48625E-4 (-1.79586 -1.17664)
$\rho = 0$	0.012E-4	0.17708E-4 (-0.142444 0.496605)
$\rho = 1$	1.14E-4	1.25475E-4 (0.922997 1.5865)

Same run with κ, ν increased by a factor 5 and μ by a factor 3 :

CDS prices	Gaussian Mapping	Monte Carlo value and 95% window
$\rho = -1$	-1.03E-4	-1.77E-4 (-2.02 -1.51)
$\rho = 0$	0.021E-4	0.143E-4 (-0.138 0.424)
$\rho = 1$	1.07E-4	1.08E-4 (0.78 1.37)

Table 1: 5y CDS prices as a function of ρ with MC simulation

The deterministic (same as $\rho = 0$) model prices with R^{BID} and R^{ASK} above are CDS^{BID} =-17.14 E-4, CDS^{ASK} =17.16 E-4, so **correlation yields an effect that is about 1/10 the bid-ask spread** (for the Gaussian mapping method see Brigo Alfonsi (2002)).

SSRD model: the impact of correlation and separable calibration

Since on CDS prices with the SSRD model correlation yields an effect that is about $1/10$ the bid-ask spread, we may calibrate the model to CDSs by assuming $\rho = 0$, i.e. by separately calibrating rates to $(\alpha, \phi(\cdot;\alpha))$ and intensities to $(\beta, \psi(\cdot,\beta))$.

We then set ρ to a desired value $\neq 0$ (ρ could be estimated historically for example).

The interest rate desk can thus provide the credit desk with a model for r independently calibrated only to the default free market, to be used together with the stochastic intensity λ the credit desk has calibrated to implied CDS "default probabilities".

SSRD model: the impact of ρ on different stylized payoffs

Although ρ has almost no impact on CDS's, allowing thus for the separate calibration above, it may have a measurable impact on products with stronger nonlinearity than CDS's (cancellable swaps, CDS options, tranches of some defaultable structures)
Examples: Consider the following terms with λ and r calibrated as before.

$$A = D(0,5y)L(4,5)\mathbf{1}_{\tau<5}, \quad B = D(0,\tau)\mathbf{1}_{\tau<5},$$

$$C = D(0,\min(\tau,5)), \quad D = D(0,5)L(4,5)\mathbf{1}_{\tau\in[4,5]},$$

These payoffs appear typically in basic credit derivatives. Traders may check the impact of correlation on different products. $L(t,T)$ is the LIBOR rate at t for maturity T.

	$\rho = -1$	$\rho = 1$	rel variation	abs variation
A	30.3672 bps	31.1962	+2.73%	+0.829
B	679.197 bps	676.208	-0.44%	-2.989
C	8207.23 bps	8209.61	+0.03%	+2.38
D	2.77376 bps	3.10889	+10.77%	+0.34

CDS Options with the SSRD model (CIR++ stochastic intensity)

$$\begin{aligned} dx_t &= k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k,\theta,\sigma,x_0), \quad 2k\theta > \sigma^2, \\ dy_t &= \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t, \; \beta = (\kappa,\mu,\nu,y_0), \; 2\kappa\mu > \nu^2, \quad dZ\,dW = \rho\,dt \\ r_t &= x_t + \phi(t;\alpha), \quad \lambda_t = y_t + \psi(t,\beta). \end{aligned}$$

In case of deterministic r and CDS-calibrated stochastic λ, **an exact closed form formula can be derived for CDS options and callable floaters** (Brigo (2004)). For stochastic r, some approximated formulas are possible.

More generally one may compute the CDS option price by means of Monte Carlo simulations, equate this MC price to our earlier Black-Scholes market Formula applied to the same CDS option at $t = 0$, and solve in $\sigma_{a,b}$.

Define implied SSRD volatility as the solution $\sigma_{a,b}^{imp}(\alpha,\beta,\rho)$ of

"MarketFormula$(\sigma_{a,b})$**= SSRD**(α,β,ρ) **Monte Carlo Option Price"**.

CDS Options with the SSRD model (CIR++ stochastic intensity)

$$
\begin{aligned}
dx_t &= k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k,\theta,\sigma,x_0), \quad 2k\theta > \sigma^2, \\
dy_t &= \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t, \;\; \beta = (\kappa,\mu,\nu,y_0), \;\; 2\kappa\mu > \nu^2, \qquad dZ\,dW = \rho\,dt \\
r_t &= x_t + \varphi(t;\alpha), \quad \lambda_t = y_t + \psi(t,\beta).
\end{aligned}
$$

Define implied SSRD volatility as the solution $\sigma_{a,b}^{imp}(\alpha,\beta,\rho)$ of **"MarketFormula**$(\sigma_{a,b})$**= SSRD**(α,β,ρ) **Monte Carlo Option Price"**.

This $\sigma_{a,b}$ is the implied volatility corresponding to the SSRD pricing model. The first numerical results we found point out the following patterns of $\sigma_{a,b}$ in terms of SSRD model parameters. For more details see Brigo and Cousot (2004). The patterns are reasonable.

Param :	$\kappa \uparrow$	$\mu \uparrow$	$\nu \uparrow$	$y_0 \uparrow$	$\rho\uparrow$
$\sigma_{a,b}^{imp}$:	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$

Conclusions and Further Research 1

For reduced form default models, we illustrated the path from constant intensity (theoretical framework) to time varying intensity (deterministic intensity models, implied CDS intensity, implied credit spread) and to stochastic intensity (credit spread volatility, CDS Options).

We have seen **explicit stochastic intensity models** like the SSRD CIR++ model, calibrated to CDS data and with CDS options formulas, and models where **the stochastic intensity is embedded in more fundamental market variables**, i.e. the market model. The SSRD model advantages are, against other explicit stochastic intensity models:

- Positive intensity and Analytical tractability
- Analytical Automatic CDS calibration
- Separability of calibration to the interest-rate and credit derivatives CDS markets (potentially useful for consistency between different trading desks)
- Analytical formula for CDS options under deterministic rates (and stochastic intensity)

We have also hinted at numerical investigation on the patterns between the SSRD CDS options implied volatility and the SSRD dynamics parameters, i.e. **a first comparison between the SSRD model and market models.**

Conclusions and Further Research 2

We have shown equivalence between CDS payoffs and Defaultable floaters payoffs, so that all models that price one product can in principle price the other one

Further research: Calibration of the β parameters in the SSRD dynamics to options data is to be investigated.

The impact of ρ (correlation interest rates / intensities) on CDS options in the SSRD model is to be investigated further.

Further work with the SSRD model includes valuing different credit derivative payoffs, deriving further analytical approximations, and study calibration and pricing for a larger variety of names. Also the extension to multi name situations, possibly via thresholds copulas, is to be investigated.

Further work with the market model includes possible extension to volatility smiles in the CDS Options markets, and the selection of some key CDS forward rates to be used as building blocks, similarly to how forward LIBOR rates are basic building blocks in market models.

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